DEDUCTIBILITY AND ANALOGY IN THE STUDY OF TRIANGLES (IV) - THE H-CEVIAN TRIANGLE AND THE H-CIRCUMCEVIAN TRIANGLE –

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Abstract. As in the first three papers with the same generic title, in this paper we propose, using logical deductibility relations and the method of analogy, to present some interesting results in Triangle Geometry. Thus, we consider a triangle ABC and the altitudes of the triangle, which intersect at the point H, called the orthocenter of the triangle and which intersect the sides of the given triangle at the points A′, B′ and C′, and the circumscribed circle of the triangle ABC at the points A_1, B_1 and C_1. Then, we will call the triangle A′B′C′ the H-cevian triangle attached to the triangle ABC and the point H, and the triangle $A_1B_1C_1$ we will call the H-circumcevian triangle attached to the triangle ABC and the point H. Using usual mathematical knowledge, valid in any triangle, but also the results presented in the first work mentioned above, we can obtain a series of very interesting geometric or trigonometric identities and inequalities, some of them very difficult to prove, synthetically. On the other hand, these new geometric or trigonometric relations introduced in certain derivable or only integrable functions, can involve a series of differential or integral identities or inequalities, particularly interesting. The work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

Keywords: deductibility, analogy, triangle, cevian, circumcevian, circle, altitude, geometric / trigonometric, identity, inequality

Introduction

The present paper is a particularization and continuation of the paper (Vălcan, 2021).

The work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

The end of a proof or a mathematical propositions which does not prove will be marked with "□".

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Preliminaries

According to what was stated above, in this paragraph, we will present the main results obtained in (Vălcan, 2021), keeping the numberings and notations there. In this sense, we consider a triangle ABC and the following definitions – see the figure below.

Definition 2.1: *A triangle inscribed in triangle ABC is called a triangle A′B′C′, the vertices of which are on the sides of triangle ABC, i.e. for which A′∈(BC), B′∈(CA) and C′∈(AB).*

Definitions 2.2: *Let ABC be a triangle and the cevians AA′, BB′, CC′ which intersects at point K, with A′∈(BC), B′∈(CA) and C′∈(AB). Also, let A_1, B_1, C_1 be the points where these cevians intersect the circle circumscribed to the triangle ABC for the second time. Then*:

- *the triangle A′B′C′ is called the K-cevian triangle attached to the triangle ABC and the point K*;
- *the triangle $A_1B_1C_1$ is called the K-circumcevian triangle attached to the triangle ABC and the point K.*

So, the K-cevian triangle attached to a triangle and the point K, is the triangle formed by the intersections with the sides of the respective triangle, of three cevians, which intersects at point K, and the K-circumcevian triangle attached to a triangle and the point K, is the triangle formed by the intersections (the second time) with the circle circumscribed to the respective triangle, of three cevians, which intersects at point K.

If [AA′], [BB′] and [CC′] are the altitudes of triangle ABC, with A′∈BC, B′∈CA and C′∈AB, then these altitudes intersect at point H – the orthocenter of triangle ABC and ΔA′B′C′ is called H-cevian triangle attached to triangle ABC and point H. If A_1, B_1, C_1 are the points where these altitudes intersect the circle circumscribing the triangle for the second time, then $\Delta A_1B_1C_1$ is called the H-circumcevian triangle attached to the triangle ABC and the point H.

We specify the fact that the H-cevian triangle is also called the orthic triangle of the triangle ABC.

We will denote by a, b, c the lengths of the sides of triangle ABC, by a′, b′, c′ that the lengths of the sides of triangle A′B′C′ and by a_1, b_1, c_1 the lengths of the sides of triangle $A_1B_1C_1$. We will also denote by S′, p′ and r′ - the area, semiperimeter and radius of the circle inscribed in the triangle A′B′C′ and with S_1, p_1 and r_1 – the area, semiperimeter and radius of the circle inscribed in the triangle $A_1B_1C_1$.

We assume that the following equalities hold:

$BA' = \alpha \cdot BC$, $CB' = \beta \cdot CA$ and $AC' = \gamma \cdot AB$. (2.1)

Then,

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A′C=(1-α)·BC, B′A=(1-β)·CA and C′B=(1-γ)·AB. (2.2)

$S'=2\cdot\alpha\cdot\beta\cdot\gamma\cdot S=2\cdot(1-\alpha)\cdot(1-\beta)\cdot(1-\gamma)\cdot S.$ (2.13)

$S'\leq\frac{S}{4}$. (2.17)

$a'^2=\gamma\cdot(1-\beta)\cdot a^2+(1-\beta)\cdot(1-\beta-\gamma)\cdot b^2+\gamma\cdot(\beta+\gamma-1)\cdot c^2;$ (2.20)

$b'^2=\alpha\cdot(\alpha+\gamma-1)\cdot a^2+(1-\gamma)\cdot(1-\alpha-\gamma)\cdot c^2+\alpha\cdot(1-\gamma)\cdot b^2;$ (2.20′)

$c'^2=(1-\alpha)\cdot(1-\alpha-\beta)\cdot a^2+\beta\cdot(\alpha+\beta-1)\cdot b^2+\beta\cdot(1-\alpha)\cdot c^2.$ (2.20″)

Regarding the S_1 area, we make it clear that, in general, it cannot be precisely determined / calculated, because this depends on several parameters. For example, if we make the following notations:

$\sphericalangle ABB'=\sphericalangle AA_1B_1 \overset{\text{not.}}{=} x$, $\sphericalangle BCC'=\sphericalangle BB_1C \overset{\text{not.}}{=} y$ and $\sphericalangle CAA'=\sphericalangle CC_1A_1 \overset{\text{not.}}{=} z$, (2.22)

then:

$\sphericalangle B'BC=\sphericalangle CC_1B_1 \overset{\text{not.}}{=} B-x$, $\sphericalangle C'CA=\sphericalangle C_1A_1A \overset{\text{not.}}{=} C-y$ and $\sphericalangle A'AB=\sphericalangle A_1B_1B \overset{\text{not.}}{=} A-z$. (2.23)

But:

$\sphericalangle C_1A_1B_1 \overset{\text{not.}}{=} A_1=C-y+x$, $\sphericalangle A_1B_1C_1 \overset{\text{not.}}{=} B_1=A-z+y$, and $\sphericalangle B_1C_1A_1 \overset{\text{not.}}{=} C_1=B-x+z$. (2.24)

Then, we obtain that:

$a_1=2\cdot R\cdot\sin(C-y+x)$, $b_1=2\cdot R\cdot\sin(A-z+y)$ and $c_1=2\cdot R\cdot\sin(B-x+z)$, (2.25′)

$p_1=\frac{a_1+b_1+c_1}{2}=R\cdot[\sin(A-z+y)+\sin(B-x+z)+\sin(C-y+x)]$

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$$=4\cdot R\cdot \cos\frac{A-z+y}{2}\cdot\cos\frac{B-x+z}{2}\cdot\cos\frac{C-y+x}{2};$$
(2.26)

$$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot\sin(A-z+y)\cdot\sin(B-x+z)\cdot\sin(C-y+x);$$
(2.27)

$$r_1=\frac{S_1}{p_1}=4\cdot R\cdot\sin\frac{A-z+y}{2}\cdot\sin\frac{B-x+z}{2}\cdot\sin\frac{C-y+x}{2}. \square$$
(2.28)

Next, we will calculate the lengths AA′, BB′ and CC′:

Proposition 2.8: *The following equalities hold*:

$AA'^2=(\alpha^2-\alpha)\cdot a^2+\alpha\cdot b^2+(1-\alpha)\cdot c^2;$
(2.29)

$BB'^2=(1-\beta)\cdot a^2+(\beta^2-\beta)\cdot b^2+\beta\cdot c^2;$
(2.29′)

$CC'^2=\gamma\cdot a^2+(1-\gamma)\cdot b^2+(\gamma^2-\gamma)\cdot c^2. \square$
(2.29″)

On the other hand, the following equalities hold:

$$A'A_1=\frac{\alpha\cdot(1-\alpha)\cdot a^2}{\sqrt{(\alpha^2-\alpha)\cdot a^2+\alpha\cdot b^2+(1-\alpha)\cdot c^2}},$$
(2.34)

$$B'B_1=\frac{\beta\cdot(1-\beta)\cdot b^2}{\sqrt{(1-\beta)\cdot a^2+(\beta^2-\beta)\cdot b^2+\beta\cdot c^2}},$$
(2.34′)

$$C'C_1=\frac{\gamma\cdot(1-\gamma)\cdot c^2}{\sqrt{\gamma\cdot a^2+(1-\gamma)\cdot b^2+(\gamma^2-\gamma)\cdot c^2}}. \square$$
(2.34″)

Applying Menelaus' Theorem in the triangle ABA′ for the transversal C′KC, we obtain:

$$\frac{KA'}{KA}=\frac{(1-\alpha)\cdot(1-\gamma)}{\gamma}=\frac{\alpha\cdot\beta}{1-\beta}.$$
(2.36)

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$\frac{KB'}{KB} = \frac{(1-\alpha)\cdot(1-\beta)}{\alpha} = \frac{\beta\cdot\gamma}{1-\gamma}$ and $\frac{KC'}{KC} = \frac{(1-\beta)\cdot(1-\gamma)}{\beta} = \frac{\alpha\cdot\gamma}{1-\alpha}$.(2.36′)

From the equalities (2.36) and (2.36′), it follows that:

$$\frac{KA'}{KA} \cdot \frac{KB'}{KB} \cdot \frac{KC'}{KC} = \frac{\alpha\cdot\beta\cdot\gamma}{(1-\alpha)\cdot(1-\beta)\cdot(1-\gamma)} \cdot \alpha\cdot\beta\cdot\gamma = \alpha\cdot\beta\cdot\gamma \le \frac{1}{8}. \square \quad (2.37)$$

From the equalities (2.40) and (2.41), by addition, we obtain that:

$$S_{\Delta BA_1C} = \frac{\alpha\cdot(1-\alpha)\cdot a^2}{AA'^2}\cdot S. \quad (2.42)$$

Analogously, obtain that:

$$S_{\Delta CB_1A} = \frac{\beta\cdot(1-\beta)\cdot b^2}{BB'^2}\cdot S \quad \text{and} \quad S_{\Delta AC_1B} = \frac{\gamma\cdot(1-\gamma)\cdot c^2}{CC'^2}\cdot S. \quad (2.42')$$

From these last three equalities, it follows that:

$$S_{\Delta BA_1C} + S_{\Delta CB_1A} + S_{\Delta AC_1B} \le \frac{S}{4}\cdot\left(\frac{a^2}{AA'^2}+\frac{b^2}{BB'^2}+\frac{c^2}{CC'^2}\right). \square \quad (2.43)$$

From here, it follows that:

$$A'A_1 = \frac{\alpha\cdot(1-\alpha)\cdot a^2}{AA'} \le \frac{a^2}{4\cdot AA'}. \quad (2.45)$$

Analogously, obtain that:

$$B'B_1 = \frac{\beta\cdot(1-\beta)\cdot b^2}{BB'} \le \frac{b^2}{4\cdot BB'} \quad \text{and} \quad C'C_1 = \frac{\lambda\cdot(1-\gamma)\cdot c^2}{CC'} \le \frac{c^2}{4\cdot CC'}. \square \quad (2.45')$$

At the end of this paragraph, we have the following results:

Proposition 2.9: *The following equalities hold*:

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$$AA_1 = \frac{\alpha \cdot b^2 + (1-\alpha) \cdot c^2}{\sqrt{(\alpha^2 - \alpha) \cdot a^2 + \alpha \cdot b^2 + (1-\alpha) \cdot c^2}}; \quad (2.46)$$

$$BB_1 = \frac{(1-\beta) \cdot a^2 + \beta \cdot c^2}{\sqrt{(1-\beta) \cdot a^2 + (\beta^2 - \beta) \cdot b^2 + \beta \cdot c^2}}; \quad (2.46')$$

$$CC_1 = \frac{\gamma \cdot a^2 + (1-\gamma) \cdot b^2}{\sqrt{\gamma \cdot a^2 + (1-\gamma) \cdot b^2 + (\gamma^2 - \gamma) \cdot c^2}}. \quad (2.46'')$$ □

2. Main results

In this paragraph we will refer to the H-cevian triangle and the H-circumcevian triangle attached to a triangle ABC and the point H – the orthocenter of the triangle ABC.

Consider the figure below, where [AA′], [BB′] and [CC′] are the heights of triangle ABC, with $A' \in BC$, $B' \in CA$ and $C' \in AB$, which intersect at point H – the orthocenter of the triangle ABC and where A_1, B_1, C_1 are the points where these heights intersect the circle circumscribing the triangle a second time. So, as shown above, $\Delta A'B'C'$ is the H-cevian triangle attached to triangle ABC and point H, and $\Delta A_1B_1C_1$ is the H-circumcevian triangle attached to triangle ABC and point H.

We specify the fact that we will distinguish two cases, which are required for our study:

a) triangle ABC is acute - see Figure 1;

b) triangle ABC is obtuse - see Figure 2.

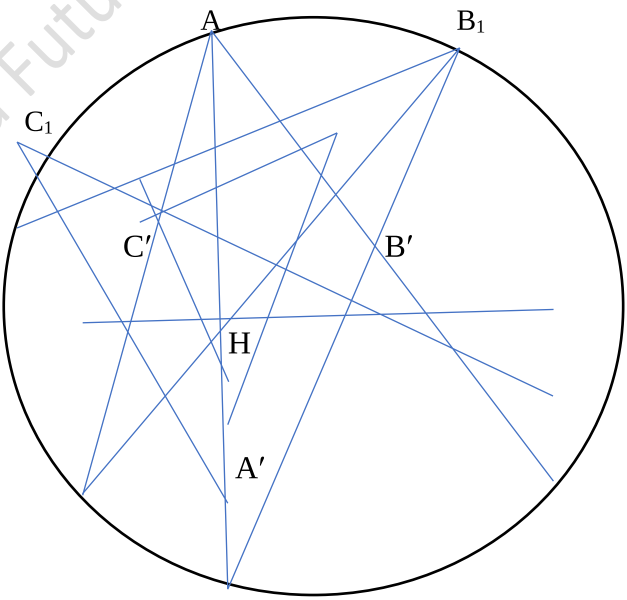

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Figure 1

We remind you that ΔA′B′C′ is also called the orthic triangle associated with triangle ABC.

According to the hypothesis, in the case of the acute-angled triangle, we obtain that:

$A'B=\frac{a\cdot\sin C\cdot\cos B}{\sin A}$, $A'C=\frac{a\cdot\sin B\cdot\cos C}{\sin A}$, $B'C=\frac{b\cdot\sin A\cdot\cos C}{\sin B}$,

$B'A=\frac{b\cdot\sin C\cdot\cos A}{\sin B}$, $C'A=\frac{c\cdot\sin B\cdot\cos A}{\sin C}$, $C'B=\frac{c\cdot\sin A\cdot\cos B}{\sin C}$.

So in this case,

$$\alpha=\frac{\sin C\cdot\cos B}{\sin A},\quad \beta=\frac{\sin A\cdot\cos C}{\sin B},\quad \gamma=\frac{\sin B\cdot\cos A}{\sin C},\qquad (3.1)$$

and, it is immediately verified that:

$$1-\alpha=\frac{\sin B\cdot\cos C}{\sin A},\quad 1-\beta=\frac{\sin C\cdot\cos A}{\sin B},\quad 1-\gamma=\frac{\sin A\cdot\cos B}{\sin C},\qquad (3.1')$$

In the case of the obtuse triangle, we obtain that:

C′A=-b·cosA, C′B=a·cosB and C′C=b·sinA (3.2)

and the analogues:

B′A=-c·cosA, B′C=a·cosC and B′B=a·sinC; (3.2′)

A′B=c·cosB, A′C=b·cosC and A′A=c·sinB. (3.2″)

Then:

1) From the equalities (2.13) and (3.1), respectively (3.2), (3.2′) and (3. 2″), we obtain that, in both situations:

$$S'=2\cdot|\cos A\cdot\cos B\cdot\cos C|\cdot S,\qquad (3.3)$$

equality, which, in the case of the acute-angled triangle, becomes:

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$S' = 2\cdot\cos A\cdot\cos B\cdot\cos C\cdot S.$

(3.3′)

Now, the inequality (2.17) is immediate, because, according to (Andrica, Jecan & Magdaș, 2019, p. 139):

$\cos A\cdot\cos B\cdot\cos C \le \frac{1}{8}.$

(i)

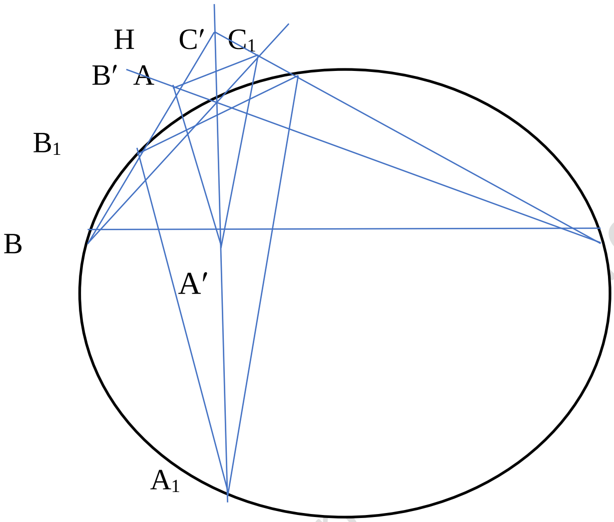

Figure 2.

2) From the equalities (2.20), (2.20′) and (2.20″), respectively (3.1), (3.1′), (3.2), (3.2′) and (3.2″), we obtain the lengths of the sides of the H-cevian triangle:

$a' = B'C' = a\cdot|\cos A| = 2\cdot R\cdot\sin A\cdot|\cos A|,$

(3.4)

$b' = A'C' = b\cdot|\cos B| = 2\cdot R\cdot\sin B\cdot|\cos B|,$

(3.4′)

$c' = A'B' = c\cdot|\cos C| = 2\cdot R\cdot\sin C\cdot|\cos C|.$

(3.4″)

So, in the case of the acute-angled triangle, the equalities hold:

$a' = B'C' = a\cdot|\cos A| = 2\cdot R\cdot\sin A\cdot|\cos A| = 2\cdot R\cdot\sin A\cdot\cos A = R\cdot\sin(2\cdot A),$

(3.5)

$b' = A'C' = b\cdot|\cos B| = 2\cdot R\cdot\sin B\cdot|\cos B| = 2\cdot R\cdot\sin B\cdot\cos B = R\cdot\sin(2\cdot B),$

(3.5′)

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$c' = A'B' = c\cdot|\cos C| = 2\cdot R\cdot \sin C\cdot|\cos C| = 2\cdot R\cdot \sin C\cdot \cos C = R\cdot \sin(2\cdot C)$.

(3.5″)

3) From the equalities (3.4), (3.4′) and (3.4″), it follows that,

$$p' = \frac{a'+b'+c'}{2} = R\cdot(\sin A\cdot|\cos A| + \sin B\cdot|\cos B| + \sin C\cdot|\cos C|),$$

(3.6)

equality, which in the case of the acute-angled triangle, see equalities (3.5), (3.5′) and (3.5″), becomes:

$$p' = R\cdot(\sin A\cdot \cos A + \sin B\cdot \cos B + \sin C\cdot \cos C) = 2\cdot R\cdot \sin A\cdot \sin B\cdot \sin C$$

$$= 16\cdot R\cdot \sin\frac{A}{2}\cdot \sin\frac{B}{2}\cdot \sin\frac{C}{2}\cdot \cos\frac{A}{2}\cdot \cos\frac{B}{2}\cdot \cos\frac{C}{2} = 4\cdot \sin\frac{A}{2}\cdot \sin\frac{B}{2}\cdot \sin\frac{C}{2}\cdot p.$$

(3.6′)

because, according to (Ţigănilă & Dumitru, 1979, p. 287), the following equality takes holds:

$$\sin(2\cdot A) + \sin(2\cdot B) + \sin(2\cdot C) = 4\cdot \sin A\cdot \sin B\cdot \sin C.$$

(3.7)

4) From the equalities (3.4), (3.4′), (3.4″), (3.3) and (3.6), it follows that,

$$r' = \frac{S'}{p'} = 2\cdot \frac{|\cos A\cdot \cos B\cdot \cos C|}{\sin A\cdot|\cos A| + \sin B\cdot|\cos B| + \sin C\cdot|\cos C|}\cdot \frac{S}{R}$$

$$= 2\cdot \frac{|\cos A\cdot \cos B\cdot \cos C|}{\sin A\cdot|\cos A| + \sin B\cdot|\cos B| + \sin C\cdot|\cos C|}\cdot \frac{r\cdot p}{R}$$

$$= 2\cdot \frac{|\cos A\cdot \cos B\cdot \cos C|}{\sin A\cdot|\cos A| + \sin B\cdot|\cos B| + \sin C\cdot|\cos C|}\cdot \frac{r\cdot 4\cdot R\cdot \cos\frac{A}{2}\cdot \cos\frac{B}{2}\cdot \cos\frac{C}{2}}{R}$$

$$= \frac{8\cdot|\cos A\cdot \cos B\cdot \cos C|\cdot \cos\frac{A}{2}\cdot \cos\frac{B}{2}\cdot \cos\frac{C}{2}}{\sin A\cdot|\cos A| + \sin B\cdot|\cos B| + \sin C\cdot|\cos C|}\cdot r;$$

(3.8)

because, according to (Ţigănilă & Dumitru, 1979, p. 287), the following equality takes holds:

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$\sin A+\sin B+\sin C=4\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$.

(3.9)

In the case of the acute-angled triangle, according to equalities (3.3′), (3.6′) and (3.9), the equalities from (3.8) become:

$$r'=\frac{S'}{p'}=2\cdot\frac{\cos A\cdot\cos B\cdot\cos C}{\sin A\cdot\cos A+\sin B\cdot\cos B+\sin C\cdot\cos C}\cdot\frac{S}{R}$$

$$=4\cdot\frac{\cos A\cdot\cos B\cdot\cos C}{\sin(2\cdot A)+\sin(2\cdot B)+\sin(2\cdot C)}\cdot\frac{r\cdot p}{R}$$

$$=4\cdot\frac{\cos A\cdot\cos B\cdot\cos C}{4\cdot\sin A\cdot\sin B\cdot\sin C}\cdot\frac{r\cdot 4\cdot R\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{R}$$

$$=\frac{\cos A\cdot\cos B\cdot\cos C}{\sin A\cdot\sin B\cdot\sin C}\cdot 4\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot r$$

$$=\frac{\cos A\cdot\cos B\cdot\cos C}{8\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}\cdot 4\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot r$$

$$=\frac{\cos A\cdot\cos B\cdot\cos C}{2\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\cdot r.$$

(3.8′)

5) On the other hand, from the equalities (3.3), (3.4), (3.4′) and (3.4″), we obtain that:

$$R'=\frac{a'\cdot b'\cdot c'}{4\cdot S'}=\frac{8\cdot R^3\cdot\sin A\cdot\sin B\cdot\sin C\cdot|\cos A\cdot\cos B\cdot\cos C|}{8\cdot|\cos A\cdot\cos B\cdot\cos C|\cdot S}$$

$$=\frac{R^3\cdot\sin A\cdot\sin B\cdot\sin C}{S}=\frac{R^3\cdot\sin A\cdot\sin B\cdot\sin C\cdot 4\cdot R}{a\cdot b\cdot c}$$

$$=\frac{4\cdot R^4\cdot\sin A\cdot\sin B\cdot\sin C}{8\cdot R^3\cdot\sin A\cdot\sin B\cdot\sin C}=\frac{R}{2}.$$

(3.10)

6) Equalities (2.22), (2.23) and (2.24) become:

a) if triangle ABC is acute-angled (see Figure 1), then:

∢A′=180°-2·A, ∢B′=180°-2·B and ∢C′=180°-2·C, (3.11)

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and:

$\measuredangle A_1=180°-2\cdot A$, $\measuredangle B_1=180°-2\cdot B$ and $\measuredangle C_1=180°-2\cdot C$; (3.12)

b) if triangle ABC is obtuse (see Figure 2), then:

$\measuredangle A'=2\cdot A-180°$, $\measuredangle B'=2\cdot B$ and $\measuredangle C'=2\cdot C$, (3.11′)

and:

$\measuredangle A_1=2\cdot A-180°$, $\measuredangle B_1=2\cdot B$ and $\measuredangle C_1=2\cdot C$. (3.12′)

Now, we notice that the equalities from (3.10) are very easily obtained using the formula:

$$R'=\frac{a'}{2\cdot \sin A'}. \quad (3.13)$$

Next we have the following results:

7) The equalities (2.25′) become:

a) if triangle ABC is acute-angled, then:

$a_1=2\cdot R\cdot \sin A_1=2\cdot R\cdot \sin(2\cdot A)$ and the analogues:

$b_1=2\cdot R\cdot \sin B_1=2\cdot R\cdot \sin(2\cdot B)$, $c_1=2\cdot R\cdot \sin C_1=2\cdot R\cdot \sin(2\cdot C)$. (3.14)

b) if triangle ABC is obtuse, then:

$a_1=2\cdot R\cdot \sin A_1=2\cdot R\cdot |\sin(2\cdot A)|$ and the analogues:

$b_1=2\cdot R\cdot \sin B_1=2\cdot R\cdot \sin(2\cdot B)$, $c_1=2\cdot R\cdot \sin C_1=2\cdot R\cdot \sin(2\cdot C)$. (3.14′)

8) The equalities (2.26) become:

a) if the triangle ABC is acute-angled, then, according to equalities (3.14) and (3.7):

$$p_1=\frac{a_1+b_1+c_1}{2}=R\cdot[\sin(2\cdot A)+\sin(2\cdot B)+\sin(2\cdot C)]=4\cdot R\cdot \sin A\cdot \sin B\cdot \sin C$$

$$=32\cdot R\cdot \sin\frac{A}{2}\cdot \sin\frac{B}{2}\cdot \sin\frac{C}{2}\cdot \cos\frac{A}{2}\cdot \cos\frac{B}{2}\cdot \cos\frac{C}{2}=8\cdot \sin\frac{A}{2}\cdot \sin\frac{B}{2}\cdot \sin\frac{C}{2}\cdot p; \quad (3.15)$$

b) if the triangle ABC is obtuse, then, according to equalities (3.14′):

$$p_1=\frac{a_1+b_1+c_1}{2}=R\cdot[-\sin(2\cdot A)+\sin(2\cdot B)+\sin(2\cdot C)]=4\cdot R\cdot \sin A\cdot \cos B\cdot \cos C$$

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$$=8\cdot R\cdot \sin\frac{A}{2}\cdot\cos\frac{A}{2}\cdot\cos B\cdot\cos C=2\cdot\frac{\sin\frac{A}{2}\cdot\cos B\cdot\cos C}{\cos\frac{B}{2}\cdot\cos\frac{C}{2}}\cdot p;$$

(3.15′)

because, it is immediately verified that:

$$-\sin(2\cdot A)+\sin(2\cdot B)+\sin(2\cdot C)=4\cdot R\cdot\sin A\cdot\cos B\cdot\cos C.$$

(3.16)

9) The equalities (2.27) become:

a) if the triangle ABC is acute-angled, then, according to equalities (3.14):

$$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=2\cdot R^2\cdot\sin(2\cdot A)\cdot\sin(2\cdot B)\cdot\sin(2\cdot C)$$

$$=16\cdot R^2\cdot\sin A\cdot\sin B\cdot\sin C\cdot\cos A\cdot\cos B\cdot\cos C=8\cdot\cos A\cdot\cos B\cdot\cos C\cdot S;$$

(3.17)

or else:

$$S_1=\frac{b_1\cdot c_1\cdot\sin A_1}{2}=2\cdot R^2\cdot\sin(2\cdot A)\cdot\sin(2\cdot B)\cdot\sin(2\cdot C)=8\cdot\cos A\cdot\cos B\cdot\cos C\cdot S;$$

or else:

$$S_1=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=8\cdot\cos A\cdot\cos B\cdot\cos C\cdot S;$$

b) if the triangle ABC is obtuse, then, according to equality (3.14′):

$$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=2\cdot R^2\cdot|\sin(2\cdot A)|\cdot\sin(2\cdot B)\cdot\sin(2\cdot C)$$

$$=-16\cdot R^2\cdot\sin A\cdot\sin B\cdot\sin C\cdot\cos A\cdot\cos B\cdot\cos C=8\cdot|\cos A|\cdot\cos B\cdot\cos C\cdot S;$$

(3.17′)

or else:

$$S_1=\frac{b_1\cdot c_1\cdot\sin A_1}{2}=-2\cdot R^2\cdot\sin(2\cdot A)\cdot\sin(2\cdot B)\cdot\sin(2\cdot C)=8\cdot|\cos A|\cdot\cos B\cdot\cos C\cdot S;$$

or else:

$$S_1=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=8\cdot|\cos A|\cdot\cos B\cdot\cos C\cdot S.$$

10) The equalities (2.28) become:

a) if the triangle ABC is acute-angled, then, according to equalities (3.17) and (3.15):

$$r_1=\frac{S_1}{p_1}=4\cdot R\cdot\sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=4\cdot R\cdot\sin A\cdot\sin B\cdot\sin C$$

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$$=32\cdot R\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=8\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot r;$$
(3.18)

b) if the triangle ABC is obtuse, then, according to equality (3.17′) and (3.15′):

$$r_1=\frac{S_1}{p_1}=4\cdot R\cdot\sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=-4\cdot R\cdot\cos A\cdot\sin B\cdot\sin C$$

$$=-16\cdot R\cdot\cos A\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=4\cdot\frac{|\cos A|\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}}\cdot r;$$
(3.18′)

or else:

$$r_1=\frac{S_1}{p_1}=\frac{8\cdot|\cos A|\cdot\cos B\cdot\cos C\cdot S\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{2\cdot\sin\frac{A}{2}\cdot\cos B\cdot\cos C\cdot p}=4\cdot\frac{|\cos A|\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}}\cdot r.$$

Now the lengths of the segments [AA′], [BB′] and [CC′] become the lengths of the altitudes of the triangle ABC.

11) From the equalities (2.29), (2.29′) and (2.29″), we obtain the equalities:

$h_a^2=4\cdot R^2\cdot\sin^2 B\cdot\sin^2 C$, $h_b^2=4\cdot R^2\cdot\sin^2 A\cdot\sin^2 C$,

and $h_c^2=4\cdot R^2\cdot\sin^2 A\cdot\sin^2 B$.

(3.19)

From equalities (3.19), we deduce that:

$h_a=2\cdot R\cdot\sin B\cdot\sin C=\frac{b\cdot c}{2\cdot R}$, $h_b=2\cdot R\cdot\sin A\cdot\sin C=\frac{a\cdot c}{2\cdot R}$,

and $h_c=2\cdot R\cdot\sin A\cdot\sin B=\frac{a\cdot b}{2\cdot R}$.

12) The equalities (2.34), (2.34′) and (2.34″) become:

$A'A_1=2\cdot R\cdot\cos B\cdot\cos C$, $B'B_1=2\cdot R\cdot|\cos A|\cdot\cos C$,

and $C'C_1=2\cdot R\cdot|\cos A|\cdot\cos B$.

(3.20)

13) The equalities (2.36) and (2.36′), become:

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$$\frac{HA'}{HA}=\frac{2\cdot R\cdot \cos B\cdot \cos C}{2\cdot R\cdot |\cos A|}=\frac{\cos B\cdot \cos C}{|\cos A|},$$ and the analogues:

$$\frac{HB'}{HB}=\frac{\cos A\cdot \cos C}{|\cos B|}$$ and $$\frac{HC'}{HC}=\frac{\cos A\cdot \cos B}{|\cos C|},$$ (3.21)

in which case (2.37) becomes:

$$\frac{HA'}{HA}\cdot\frac{HB'}{HB}\cdot\frac{HC'}{HC}=\frac{\cos B\cdot \cos C}{|\cos A|}\cdot\frac{\cos A\cdot \cos C}{|\cos B|}\cdot\frac{\cos A\cdot \cos B}{|\cos C|}=$$

$=|\cos A\cdot\cos B\cdot\cos C|\le \frac{1}{8}$, (3.22)

according to inequality (i), above, since:

$HA'=A'A_1=2\cdot R\cdot\cos B\cdot\cos C$, $HB'=B'B_1=2\cdot R\cdot|\cos A|\cdot\cos C$,
and $HC'=C'C_1=2\cdot R\cdot|\cos A|\cdot\cos B$,
(3.23)

and:

$HA=|HA-AA'|=2\cdot R\cdot|\cos A|$, $HB=|HB\pm BB'|=2\cdot R\cdot|\cos B|$,
and $HC=|HC\pm CC'|=2\cdot R\cdot|\cos C|$.
(3.24)

14) The equalities (2.42) and (2.42′), become:

$S^{\Delta BA_1C}=\text{ctg}B\cdot\text{ctg}C\cdot S$, $S^{\Delta CB_1A}=|\text{ctg}A|\cdot\text{ctg}C\cdot S$

and $S^{\Delta AC_1B}=|\text{ctg}A|\cdot\text{ctg}B\cdot S$,
(3.25)

in which case the inequality (2.43) becomes:

$$\text{ctg}B\cdot\text{ctg}C+|\text{ctg}A|\cdot\text{ctg}C+|\text{ctg}A|\cdot\text{ctg}B\le\frac{1}{4}\cdot\left(\frac{\sin^2 A}{\sin^2 B\cdot\sin^2 C}+\frac{\sin^2 B}{\sin^2 A\cdot\sin^2 C}+\frac{\sin^2 C}{\sin^2 A\cdot\sin^2 B}\right)$$ (3.26)

which is easy to verify, because:

$$\text{ctg}B\cdot\text{ctg}C\le\frac{1}{4}\cdot\frac{\sin^2 A}{\sin^2 B\cdot\sin^2 C}$$ and the analogues:

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$\text{ctgA}\cdot\text{ctgC}\le\frac{1}{4}\cdot\frac{\sin^2 B}{\sin^2 A\cdot\sin^2 C}$ and $\text{ctgA}\cdot\text{ctgB}\le\frac{1}{4}\cdot\frac{\sin^2 C}{\sin^2 A\cdot\sin^2 B}$. (3.27)

15) Relations (2.45) and (2.45′) become:

$A'A_1=2\cdot R\cdot\cos B\cdot\cos C\le\frac{R}{2}\cdot\frac{\sin^2 A}{\sin B\cdot\sin C}$, $B'B_1=2\cdot R\cdot|\cos A|\cdot\cos C\le\frac{R}{2}\cdot\frac{\sin^2 B}{\sin A\cdot\sin C}$

and $C'C_1=2\cdot R\cdot|\cos A|\cdot\cos B\le\frac{R}{2}\cdot\frac{\sin^2 C}{\sin A\cdot\sin B}$. (3.28)

16) The equalities (2.46), (2.46′) and (2.46″) become:

a) if the triangle ABC is acute-angled, then, according to equalities (3.19) and (3.20):

$AA_1=2\cdot R\cdot\cos(B-C)$ and the analogues: $BB_1=2\cdot R\cdot\cos(C-A)$; $CC_1=2\cdot R\cdot\cos(A-B)$; (3.29)

b) if the triangle ABC is obtuse, then, according to the same equalities (3.19) and (3.20):

$AA_1=2\cdot R\cdot\cos(B-C)$ and the analogues: $BB_1=2\cdot R\cdot|\cos(C+A)|$; $CC_1=2\cdot R\cdot|\cos(B+A)|$. (3.29′)

The following remarks is required here:

1. The equalities from (3.11), (3.12), (3.4) and others can also be obtained otherwise. Indeed, the quadrilaterals BA′HC′, CB′HA′ and AC′HB′ are inscribed. So,

$$\sphericalangle HA'C'=\sphericalangle HBC'=\sphericalangle HBA=\frac{\pi}{2}-A=\sphericalangle ACC'=\sphericalangle B'CH=\sphericalangle HA'B'.$$

So, [A′H is the angle bisector of A′ and:

$$\sphericalangle B'A'C'\overset{\text{not.}}{=}A'=\pi-2\cdot A.$$

Analogously, we obtain that [B′H is the bisector of angle B′ and [C′H is the bisector of angle C′. Furthermore,

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$\sphericalangle A'B'C' \overset{\text{not.}}{=} B'=\pi-2\cdot B$ and $\sphericalangle B'C'A' \overset{\text{not.}}{=} C'=\pi-2\cdot C$.

It follows that:

$\sphericalangle BA'C'=\sphericalangle CA'B'=A$, $\sphericalangle CB'A'=\sphericalangle AB'C'=B$ and $\sphericalangle AC'B'=\sphericalangle BC'A'=C$.

In the right triangle $BB'A$,

$\cos A=\dfrac{AB'}{AB}$, so:

$AB'=c\cdot\cos A$.

Now, applying the law of sine in $\Delta AB'C'$, we obtain that:

$$\frac{B'C'}{\sin A}=\frac{AB'}{\sin(AC'B')};$$

so:

$$B'C' \overset{\text{not.}}{=} a'=\frac{c\cdot\sin A\cdot\cos A}{\sin C}=\frac{a\cdot\sin A\cdot\cos A}{\sin A}=a\cdot\cos A=R\cdot\sin(2\cdot A).$$

Analogously, we obtain that:

$C'A' \overset{\text{not.}}{=} b'=b\cdot\cos B=R\cdot\sin(2\cdot B)$ and $A'B' \overset{\text{not.}}{=} c'=c\cdot\cos C=R\cdot\sin(2\cdot C)$.

2. Now, applying the law of sines to triangle $A'B'C'$, we obtain that:

$$\frac{a'}{\sin A'}=2\cdot R',$$

from which follows the equality (3.10):

$$R'=\frac{R}{2}.$$

3. For S' - in the case of the acute-angled triangle, we also have the equality:

$$S'=\frac{R^2}{2}\cdot\sin(2\cdot A)\cdot\sin(2\cdot B)\cdot\sin(2\cdot C),$$

in which case we obtain the inequality:

$$S'\le\frac{3\cdot\sqrt{3}}{16}\cdot R^2,$$

since, it is immediately verified that:

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$\sin(2\cdot A)\cdot\sin(2\cdot B)\cdot\sin(2\cdot C)\le\frac{3\cdot\sqrt{3}}{8}$.

4. For p′ - in the case of the acute-angled triangle, we also have the equalities:

$p'=\frac{R}{2}\cdot[\sin(2\cdot A)+\sin(2\cdot B)+\sin(2\cdot C)]=2\cdot R\cdot\sin A\cdot\sin B\cdot\sin C$,

in which case we obtain the inequality:

$p'\le\frac{3\cdot\sqrt{3}}{4}\cdot R$,

because, it is immediately verified that (Andrica, Jecan & Magdaş, 2019, p. 139):

$\sin A\cdot\sin B\cdot\sin C\le\frac{3\cdot\sqrt{3}}{8}$.

5. For r′ - in the case of the acute-angled triangle, we also have the equalities:

$r'=\frac{S'}{p'}=\frac{2\cdot S\cdot\cos A\cdot\cos B\cdot\cos C}{2\cdot R\cdot\sin A\cdot\sin B\cdot\sin C}=\frac{S}{R}\cdot ctgA\cdot ctgB\cdot ctgC=\frac{a\cdot b\cdot c}{4\cdot R^2}\cdot ctgA\cdot ctgB\cdot ctgC$

$=\frac{2\cdot R\cdot\sin A\cdot 2\cdot R\cdot\sin B\cdot 2\cdot R\cdot\sin C}{4\cdot R^2}\cdot ctgA\cdot ctgB\cdot ctgC=2\cdot R\cdot\cos A\cdot\cos B\cdot\cos C$;

or else:

$r'=4\cdot R'\sin\frac{A'}{2}\cdot\sin\frac{B'}{2}\cdot\sin\frac{C'}{2}=4\cdot\frac{R}{2}\cdot\sin\frac{\pi-2\cdot A}{2}\cdot\sin\frac{\pi-2\cdot B}{2}\cdot\sin\frac{\pi-2\cdot C}{2}$

$=2\cdot R\cdot\sin\left(\frac{\pi}{2}-A\right)\cdot\sin\left(\frac{\pi}{2}-B\right)\cdot\sin\left(\frac{\pi}{2}-C\right)=2\cdot R\cdot\cos A\cdot\cos B\cdot\cos C$,

in which case we obtain the inequality:

$r'\le\frac{1}{4}\cdot R$,

because, it is immediately verified that (Andrica, Jecan & Magdaş, 2019, p. 139):

$\cos A\cdot\cos B\cdot\cos C\le\frac{1}{8}$.

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6. Equalities (3.4), (3.4′) and (3.4″) can also be obtained by applying the law of cosines in the triangle AB′C′ and taking into account the same theorem in the triangle ABC.
7. Even though the expressions for the lengths of the sides of the triangle A′B′C′ are very simple, it can still be shown that: (Chirciu, 2015, p. 96)

$a'+b'+c' \le \frac{3\cdot\sqrt{3}}{2}\cdot R$ and $a_1+b_1+c_1 \le 3\cdot\sqrt{3}\cdot R$. (3.30)

Hint: One uses the equality (3.7) and then the inequality:

$\sin A\cdot\sin B\cdot\sin C \le \frac{3\cdot\sqrt{3}}{8}$. (Andrica, Jecan & Magdaș, 2019, p. 139)

8. Equality (3.2) can also be obtained as follows; we note that:

$S_{\Delta AC'B'} = \frac{AC'\cdot AB'}{2}\cdot\sin A = \frac{1}{2}\cdot\frac{c\cdot\sin B\cdot\cos A}{\sin C}\cdot\frac{b\cdot\sin C\cdot\cos A}{\sin B}\cdot\sin A$

$=\cos^2 A\cdot\frac{b\cdot c\cdot\sin A}{2} = \cos^2 A\cdot S.$ (3.31)

Analogously, we obtain that:

$S_{\Delta BA'C'} = \cos^2 B\cdot S$ and $S_{\Delta CB'A'} = \cos^2 C\cdot S$. (3.31′)

From these last three equalities result:

$S' = S_{\Delta A'B'C'} = [1-\cos^2 A-\cos^2 B-\cos^2 C]\cdot S$

$= 2\cdot|\cos A\cdot\cos B\cdot\cos C|\cdot S.$ (3.2)

9. Of course, using the above results, other interesting results regarding S′ can be obtained; for example: (Chirciu, 2019, p. 95)

$\frac{S}{S'} \le \frac{1}{8}$. (3.32)

10. Using the equalities in (3.19), we can show interesting equalities with the heights of triangle ABC. Thus, the following equalities hold:

a) $h_a+h_b+h_c = \frac{p^2+r^2+4\cdot R\cdot r}{2\cdot R}$; (3.33)

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b) $h_a \cdot h_b + h_b \cdot h_c + h_a \cdot h_c = \frac{2 \cdot r \cdot p^2}{R}$; (3.34)

c) $h_a \cdot h_b \cdot h_c = \frac{2 \cdot r^2 \cdot p^2}{R}$; (3.35)

d) $h_a^2 + h_b^2 + h_c^2 = \frac{p^4 + 2 \cdot r^2 \cdot p^2 + r^2 \cdot (4 \cdot R + r)}{R^2}$; (3.36)

e) $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$; (3.37)

f) $\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} = \frac{p^2 - r^2 - 4 \cdot R \cdot r}{2 \cdot r^2 \cdot p^2}$; (3.38)

g) $\frac{1}{h_a \cdot h_b} + \frac{1}{h_b \cdot h_c} + \frac{1}{h_c \cdot h_a} = \frac{p^2 + r^2 + 4 \cdot R \cdot r}{4 \cdot r^2 \cdot p^2}$. (3.39)

We leave the verification of these equalities to the reader who is attentive and interested in such matters.

11. Now, we can obtain very interesting inequalities. For example,

a) $h_a + h_b + h_c \le 4 \cdot R + r$; (Chirciu, 2015, pag. 13)

b) $h_a \cdot h_b + h_b \cdot h_c + h_a \cdot h_c \le 4 \cdot S \cdot \sqrt{3}$; (Chirciu, 2019, p. 53)

c) $h_a \cdot h_b \cdot h_c \le S \cdot \sqrt[4]{27 \cdot S^2}$; (Chirciu, 2015, p. 56)

d) $h_a^2 + h_b^2 + h_c^2 \ge 27 \cdot r^2$; (Chirciu, 2019, p. 51)

e) $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \ge \frac{3}{\sqrt[4]{3 \cdot S^2}}$; (Chirciu, 2015, p. 57)

f) $\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} \ge \frac{1}{3 \cdot r^2}$; (Chirciu, 2015, p. 13)

g) $\frac{1}{h_a \cdot h_b} + \frac{1}{h_b \cdot h_c} + \frac{1}{h_c \cdot h_a} \ge \frac{\sqrt{3}}{S}$. (Chirciu, 2019, pag. 60)

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Hint: a) The following inequality is immediately verified:

$\frac{p^2+r^2+4\cdot R\cdot r}{2\cdot R}$ ≤4·R+r.

(3.40)

b) Gerresten's inequality and Euler's inequality are used, i.e.:

$16\cdot R\cdot r-5\cdot r^2\le p2\le 4\cdot R^2+4\cdot R\cdot r+3\cdot r^2$ and $R\ge 2\cdot r$.

(3.41)

c) The inequality is equivalent to:

$\sqrt[3]{\frac{2\cdot S^2}{R}}\le \sqrt[4]{3}\cdot\sqrt{S}$, that is: $16\cdot S^2\le 27\cdot R^4$.

The last inequality follows from Mitrinovič's inequality and Euler's inequality, i.e.:

$3\cdot r\cdot\sqrt{3}\le p\le\frac{3\cdot R\cdot\sqrt{3}}{2}$ and $R\ge 2\cdot r$.

(3.42)

d) The same reasoning as point c) is used.

e) The inequality is equivalent to:

$3\cdot r\le\sqrt[4]{3\cdot S^2}$, that is:

$81\cdot r^4\le 3\cdot r^2\cdot p^2$.

The last inequality is an immediate consequence of Mitrinovič's inequality.

f) The known relations are used:

$3\cdot(a^2+b^2+c^2)\ge(a+b+c)^2$, $a\cdot h_a=2\cdot S$ and

$S=r\cdot p$.

g) The equality (3.37) and the following algebraic inequality are used,

$x\cdot y+y\cdot z+z\cdot x\ge 3\cdot\sqrt{x\cdot y\cdot z}$,

valid for any x, y, z>0, with x+y+z=3.

12. From, equalities (3.24) and due to the fact that, according to (M. Chirciu, 2019, p. 45),,

$\cos A+\cos B+\cos A\le\frac{3}{2}$,

it follows that:

$HA+HB+HC\le 3\cdot R$,

and, due to inequality (i), it follows that:

$HA\cdot HB\cdot HC\le R^3$.

13. From, equalities (3.6) and (3.6′), due to the fact that, according to (Andrica, Jecan & Magdaș, 2019, p. 142),

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$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\leq\frac{1}{8}$,

it follows that:

$p'\leq\frac{p}{2}$ and $p_1\leq p$.

14. Because, according to (Chriciu, 2015, p. 86):

$\cos A\cdot\cos B+\cos B\cdot\cos C+\cos C\cdot\cos A\leq\frac{5}{4}-\frac{r}{R}$,

from equalities (3.23), it follows that:

$HA'+HB'+HC'\leq\frac{1}{2}\cdot(5\cdot R-4\cdot r)$.

15. Because, according to (Chriciu, 2015, p. 87):

$\frac{1}{\cos A\cdot\cos B}+\frac{1}{\cos B\cdot\cos C}+\frac{1}{\cos C\cdot\cos A}\geq\frac{6\cdot R}{r}\geq 12$,

from equalities (3.23), it follows that:

$\frac{1}{HA'}+\frac{1}{HB'}+\frac{1}{HC'}\geq\frac{3}{r}$.

16. The following double inequality holds:

$6\cdot r\leq AH+BH+CH\leq 3\cdot R$.

Hint: It is easily shown that:

$AH+BH+CH=2\cdot R\cdot\left(1+\frac{r}{R}\right)=2\cdot R+2\cdot r$

and then Euler's Inequality is used:

$R\geq 2\cdot r$.

17. Because, according to (Chriciu, 2015, p. 128) and (Andrica, Jecan & Magdaș, 2019, p. 138):

$3\cdot(\cos A\cdot\cos B+\cos B\cdot\cos C+\cos C\cdot\cos A)\leq\sin^2A+\sin^2B+\sin^2C\leq\frac{9}{4}$,

from equalities (3.23), it follows that:

$HA'+HB'+HC'\leq\frac{9}{2}\cdot R$.

18. Because, according to (Chriciu, 2019, p. 45):

$\cos A+\cos B+\cos C\leq\frac{3}{2}$,

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from equalities (3.24), it follows that:
$HA+HB+HC \le 3 \cdot R$.

19. Because, according to (Chriciu, 2015, p. 23):
$ctgA \cdot ctgB+ctgB \cdot ctgC+ctgC \cdot ctgA=1$,
from equalities (3.25), it follows that:

$S^{\Delta BA_1C}+S^{\Delta CB_1A}+S^{\Delta AC_1B}=S$.

20. Because, according to (Chriciu, 2015, p. 56):
$$\frac{\cos A \cdot \cos B}{\cos C}+\frac{\cos B \cdot \cos C}{\cos A}+\frac{\cos C \cdot \cos A}{\cos B} \ge \frac{3}{2},$$
from equalities (3.24), it follows that:
$$\frac{HA'}{HA}+\frac{HB'}{HB}+\frac{HC'}{HC} \ge \frac{3}{2}.$$

21. The following inequality holds:
$HA \cdot tgA+HB \cdot tgB+HC \cdot tgC \le HA_1 \cdot tgA+HB_1 \cdot tgB+HC_1 \cdot tgC$.
Hint: It is shown, very easily, that:
$HA \cdot tgA+HB \cdot tgB+HC \cdot tgC=2 \cdot p$ and $HA_1 \cdot tgA+HB_1 \cdot tgB+HC_1 \cdot tgC \ge 2 \cdot p$.

22. The following double inequality holds:
$$36 \cdot r^2 \le h_a \cdot HA+h_b \cdot HB+h_c \cdot HC \le \frac{9}{2} \cdot R^2.$$
Hint: It is shown, very easily, that:
$h_a \cdot HA+h_b \cdot HB+h_c \cdot HC=p^2-r^2-4 \cdot R \cdot r$,
and then Gerretsen's inequality is used:
$16 \cdot R \cdot r-5 \cdot r^2 \le p^2 \le p^2 \le 4 \cdot R^2+4 \cdot R \cdot r+3 \cdot r^2$.

3. Conclusions and recommendations

So, we can study, in the general way, using logical deductibility and the method of analogy, H-cevian triangles and H-circumcevian triangles. About the principles underlying these two methods and their application in the didactic act, see (Vălcan, 2013).

Of course, and in this case, as I have already stated in (Vălcan, 2022, 2023), the reader attentive and interested in these matters, using usual mathematical knowledge, valid in any triangle, such as those presented in (Andrica, Jecan & Magdaș, 2019), (Chirciu, 2014, 2015), (Coța et al 1982), (Pătrașcu & Smarandache, 2020) and / or (Țigănilă & Dumitru, 1979), can obtain a series of other very interesting geometric or trigonometric identities and inequalities, some of them very difficult to prove, synthetically. On the other hand, all these geometric or trigonometric relations introduced in certain derivable or only integrable functions

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can lead to a series of differential or integral identities or inequalities, particularly interesting, such as those presented in (Stănescu, 2015).

As I stated at the beginning, the work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

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