DEDUCTIBILITY AND ANALOGY IN THE STUDY OF TRIANGLES (III) - the g-cevian triangle and the g-circumcevian triangle

Teodor Dumitru Vălcan
„Babeș-Bolyai” University, Cluj-Napoca

Abstract. As in the first paper with the same generic title, in this paper we propose, using logical deductibility relations and the method of analogy, to present some interesting results in Triangle Geometry. Thus, we consider a triangle ABC and the interior bisectors of the angles of the triangle, which intersect at point I and which intersect the sides of the given triangle at points A′, B′ and C′, and the circumscribed circle of triangle ABC at A_1, B_1 and C_1. Then, we will call the triangle A′B′C′ the I-cevian triangle attached to the triangle ABC and the point I, and the triangle $A_1B_1C_1$ we will call the I-circumcevian triangle attached to the triangle ABC and the point I. Using usual mathematical knowledge, valid in any triangle, but also the results presented in the first work mentioned above, we can obtain a series of very interesting geometric or trigonometric identities and inequalities, some of them very difficult to prove, synthetically. On the other hand, these new geometric or trigonometric relations introduced in certain derivable or only integrable functions, can involve a series of differential or integral identities or inequalities, particularly interesting. The work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

Keywords: deductibility, analogy, triangle, cevian, circumcevian, circle, medians, geometric / trigonometric, identity, inequality

Specifications

The present paper is a particularization and continuation of the paper (Vălcan, 2021).

The work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

The end of a proof or a mathematical propositions which does not prove will be marked with "□".

Preliminaries

According to what was stated above, in this paragraph, we will present the main results obtained in (Vălcan, 2021), keeping the numberings and notations there. In this sense, we consider a triangle ABC and the following definitions – see the figure below.

Definition 2.1: *A triangle inscribed in triangle ABC is called a triangle A′B′C′, the vertices of which are on the sides of triangle ABC, i.e. for which $A'\in(BC)$, $B'\in(CA)$ and $C'\in(AB)$.*

Definitions 2.2: *Let ABC be a triangle and the cevians AA′, BB′, CC′ which intersects at point K, with $A'\in(BC)$, $B'\in(CA)$ and $C'\in(AB)$. Also, let A_1, B_1, C_1 be the points where these cevians intersect the circle circumscribed to the triangle ABC for the second time. Then*:

- *the triangle A′B′C′ is called the K-cevian triangle attached to the triangle ABC and the point K*;
- *the triangle $A_1B_1C_1$ is called the K-circumcevian triangle attached to the triangle ABC and the point K.*

So, the K-cevian triangle attached to a triangle and the point K, is the triangle formed by the intersections with the sides of the respective triangle, of three cevians, which intersects at point K, and the K-circumcevian triangle attached to a triangle and the point K, is the triangle formed by the intersections (the second time) with the circle circumscribed to the respective triangle, of three cevians, which intersects at point K.

If [AA′, [BB′ and [CC′ are the medians of triangle ABC, with $A'\in(BC)$, $B'\in(CA)$ and $C'\in(AB)$, then these medians intersect at point G, called *geometric centroid* / *center of mass* or else *center of gravity* of triangle ABC and $\Delta A'B'C'$ is called the G-cevian triangle attached to triangle ABC and point G. If A_1, B_1, C_1 are the points where these triangle medians intersects the circle circumscribed by the triangle a second time, then $\Delta A_1B_1C_1$ is called the G-circumcevian triangle attached to the triangle ABC and point G.

We specify the fact that the G-cevian triangle is also called the medial triangle of the triangle ABC. We will denote by a, b, c the lengths of the sides of triangle ABC, by a′, b′, c′ that the lengths of the sides of triangle A′B′C′ and by a_1, b_1, c_1 the lengths of the sides of triangle $A_1B_1C_1$. We will also denote by S′, p′ and r′ - the area, semiperimeter and radius of the circle inscribed in the triangle A′B′C′ and with S_1, p_1 and r_1 – the area, semiperimeter and radius of the circle inscribed in the triangle $A_1B_1C_1$.

We assume that the following equalities hold:

$$BA'=\alpha\cdot BC,\qquad CB'=\beta\cdot CA\quad\text{and}\quad AC'=\gamma\cdot AB. \tag{2.1}$$

Then,

$$A'C=(1-\alpha)\cdot BC,\qquad B'A=(1-\beta)\cdot CA\quad\text{and}\quad C'B=(1-\gamma)\cdot AB. \tag{2.2}$$

$$S'=2\cdot\alpha\cdot\beta\cdot\gamma\cdot S=2\cdot(1-\alpha)\cdot(1-\beta)\cdot(1-\gamma)\cdot S. \tag{2.13}$$

$$S'\leq\frac{S}{4}\,. \tag{2.17}$$

$$a'^2=\gamma\cdot(1-\beta)\cdot a^2+(1-\beta)\cdot(1-\beta-\gamma)\cdot b^2+\gamma\cdot(\beta+\gamma-1)\cdot c^2; \tag{2.20}$$

$$b'^2=\alpha\cdot(\alpha+\gamma-1)\cdot a^2+(1-\gamma)\cdot(1-\alpha-\gamma)\cdot c^2+\alpha\cdot(1-\gamma)\cdot b^2; \tag{2.20$'$}$$

$$c'^2=(1-\alpha)\cdot(1-\alpha-\beta)\cdot a^2+\beta\cdot(\alpha+\beta-1)\cdot b^2+\beta\cdot(1-\alpha)\cdot c^2. \tag{2.20$''$}$$

Regarding the S_1 area, we make it clear that, in general, it cannot be precisely determined / calculated, because this depends on several parameters. For example, if we make the following notations:

$$\measuredangle ABB'=\measuredangle AA_1B_1 \overset{not.}{=} x, \qquad \measuredangle BCC'=\measuredangle BB_1C \overset{not.}{=} y$$

and

$$\measuredangle CAA'=\measuredangle CC_1A_1 \overset{not.}{=} z, \tag{2.22}$$

then:

$$\measuredangle B'BC=\measuredangle CC_1B_1 \overset{not.}{=} B-x, \qquad \measuredangle C'CA=\measuredangle C_1A_1A \overset{not.}{=} C-y$$

and

$$\measuredangle A'AB=\measuredangle A_1B_1B \overset{not.}{=} A-z. \tag{2.23}$$

But:

$$\measuredangle C_1A_1B_1 \overset{not.}{=} A_1=C-y+x, \qquad \measuredangle A_1B_1C_1 \overset{not.}{=} B_1=A-z+y,$$

and

$$\measuredangle B_1C_1A_1 \overset{not.}{=} C_1=B-x+z. \tag{2.24}$$

Then, we obtain that:

$$a_1=2\cdot R\cdot\sin(C-y+x), \quad b_1=2\cdot R\cdot\sin(A-z+y) \text{ and } \quad c_1=2\cdot R\cdot\sin(B-x+z), \tag{2.25$'$}$$

$$p_1=\frac{a_1+b_1+c_1}{2}=R\cdot[\sin(A-z+y)+\sin(B-x+z)+\sin(C-y+x)]$$

$$=4\cdot R\cdot\cos\frac{A-z+y}{2}\cdot\cos\frac{B-x+z}{2}\cdot\cos\frac{C-y+x}{2}\,; \tag{2.26}$$

$$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot\sin(A-z+y)\cdot\sin(B-x+z)\cdot\sin(C-y+x); \tag{2.27}$$

$$r_1=\frac{S_1}{p_1}=4\cdot R\cdot\sin\frac{A-z+y}{2}\cdot\sin\frac{B-x+z}{2}\cdot\sin\frac{C-y+x}{2}\,.\ \square \tag{2.28}$$

Next, we will calculate the lengths AA', BB' and CC':

Proposition 2.8: *The following equalities hold*:

$$AA'^2=(\alpha^2-\alpha)\cdot a^2+\alpha\cdot b^2+(1-\alpha)\cdot c^2; \tag{2.29}$$

$$BB'^2=(1-\beta)\cdot a^2+(\beta^2-\beta)\cdot b^2+\beta\cdot c^2; \tag{2.29$'$}$$

$$CC'^2=\gamma\cdot a^2+(1-\gamma)\cdot b^2+(\gamma^2-\gamma)\cdot c^2.\ \square \tag{2.29$''$}$$

On the other hand, the following equalities hold:

$$A'A_1=\frac{\alpha\cdot(1-\alpha)\cdot a^2}{\sqrt{(\alpha^2-\alpha)\cdot a^2+\alpha\cdot b^2+(1-\alpha)\cdot c^2}}, \quad (2.34)$$

$$B'B_1=\frac{\beta\cdot(1-\beta)\cdot b^2}{\sqrt{(1-\beta)\cdot a^2+(\beta^2-\beta)\cdot b^2+\beta\cdot c^2}}, \quad (2.34')$$

$$C'C_1=\frac{\gamma\cdot(1-\gamma)\cdot c^2}{\sqrt{\gamma\cdot a^2+(1-\gamma)\cdot b^2+(\gamma^2-\gamma)\cdot c^2}}. \ \square \quad (2.34'')$$

Applying Menelaus' Theorem in the triangle ABA′ for the transversal C′KC, we obtain:

$$\frac{KA'}{KA}=\frac{(1-\alpha)\cdot(1-\gamma)}{\gamma}=\frac{\alpha\cdot\beta}{1-\beta}.$$

(2.36)

$$\frac{KB'}{KB}=\frac{(1-\alpha)\cdot(1-\beta)}{\alpha}=\frac{\beta\cdot\gamma}{1-\gamma} \text{ and } \frac{KC'}{KC}=\frac{(1-\beta)\cdot(1-\gamma)}{\beta}=\frac{\alpha\cdot\gamma}{1-\alpha}. \quad (2.36')$$

From the equalities (2.36) and (2.36′), it follows that:

$$\frac{KA'}{KA}\cdot\frac{KB'}{KB}\cdot\frac{KC'}{KC}=\frac{\alpha\cdot\beta\cdot\gamma}{(1-\alpha)\cdot(1-\beta)\cdot(1-\gamma)}\cdot\alpha\cdot\beta\cdot\gamma=\alpha\cdot\beta\cdot\gamma\leq\frac{1}{8}. \ \square$$

(2.37)

From the equalities (2.40) and (2.41), by addition, we obtain that:

$$S_{\Delta BA_1C}=\frac{\alpha\cdot(1-\alpha)\cdot a^2}{AA'^2}\cdot S. \quad (2.42)$$

Analogously, obtain that:

$$S_{\Delta CB_1A}=\frac{\beta\cdot(1-\beta)\cdot b^2}{BB'^2}\cdot S \quad \text{and} \quad S_{\Delta AC_1B}=\frac{\gamma\cdot(1-\gamma)\cdot c^2}{CC'^2}\cdot S. \quad (2.42')$$

From these last three equalities, it follows that:

$$S_{\Delta BA_1C}+S_{\Delta CB_1A}+S_{\Delta AC_1B}\leq\frac{S}{4}\cdot\left(\frac{a^2}{AA'^2}+\frac{b^2}{BB'^2}+\frac{c^2}{CC'^2}\right). \ \square \quad (2.43)$$

From here, it follows that:

$$A'A_1=\frac{\alpha\cdot(1-\alpha)\cdot a^2}{AA'}\leq\frac{a^2}{4\cdot AA'}. \quad (2.45)$$

Analogously, obtain that:

$$B'B_1=\frac{\beta\cdot(1-\beta)\cdot b^2}{BB'}\le\frac{b^2}{4\cdot BB'} \quad\text{and}\quad C'C_1=\frac{\lambda\cdot(1-\gamma)\cdot c^2}{CC'}\le\frac{c^2}{4\cdot CC'}. \square \tag{2.45'}$$

At the end of this paragraph, we have the following results:

Proposition 2.9: *The following equalities hold*:

$$AA_1=\frac{\alpha\cdot b^2+(1-\alpha)\cdot c^2}{\sqrt{(\alpha^2-\alpha)\cdot a^2+\alpha\cdot b^2+(1-\alpha)\cdot c^2}}; \tag{2.46}$$

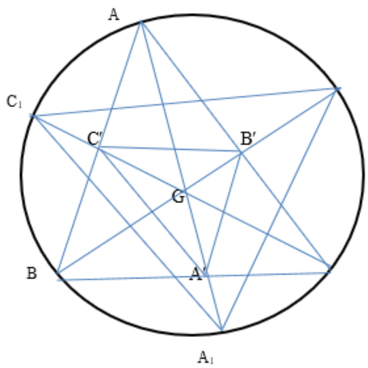
$$BB_1=\frac{(1-\beta)\cdot a^2+\beta\cdot c^2}{\sqrt{(1-\beta)\cdot a^2+(\beta^2-\beta)\cdot b^2+\beta\cdot c^2}}; \tag{2.46'}$$

$$CC_1=\frac{\gamma\cdot a^2+(1-\gamma)\cdot b^2}{\sqrt{\gamma\cdot a^2+(1-\gamma)\cdot b^2+(\gamma^2-\gamma)\cdot c^2}}. \square \tag{2.46''}$$

Main results

In this paragraph we will refer to the G-cevian triangle and the G-circumcevian triangle attached to a triangle ABC and the point G – the triangle centroid of this triangle ABC.

Consider the figure below, where [AA′, [BB′ and [CC′ are the medians of triangle ABC, with $A'\in(BC)$, $B'\in(CA)$ and $C'\in(AB)$, which intersect at point G – the triangle centroid of triangle ABC and where A_1, B_1, C_1 are the points where these medians intersect the circumscribed circle of the triangle for the second time. So, according to the above, $\Delta A'B'C'$ is the G-cevian triangle attached to triangle ABC and point G, and $\Delta A_1B_1C_1$ is the G-circumcevian triangle attached to triangle ABC and point G - see the figure below.



We remind you that $\Delta A'B'C'$ is also called the medial triangle associated with triangle ABC.

According to the hypothesis, we obtain that:

$A'B=\frac{a}{2}=A'C,$ $B'C=\frac{b}{2}=B'A,$ $C'A=\frac{c}{2}=C'B.$

So, in this case,

$$\alpha=\beta=\gamma=\frac{1}{2}. \tag{3.1}$$

Then:

1) From the equalities (2.13) and (3.1), we obtain that:

$$S'=\frac{1}{4}\cdot S. \tag{3.2}$$

So, the inequality (2.17) becomes equality.

2) From the equalities (2.20), (2.20′) and (2.20″), we obtain the lengths of the sides of the G-cevian triangle:

$$a'=B'C'=\frac{a}{2}, \tag{3.3}$$

$$b'=A'C'=\frac{b}{2}, \tag{3.3′}$$

$$c'=A'B'=\frac{c}{2}. \tag{3.3″}$$

The equalities (3.3), (3.3′) and (3.3″) can also be obtained from the characterization theorem of the middle line in the triangle.

From the equalities (3.3), (3.3′) and (3.3″) it follows that:

$$p'=\frac{a'+b'+c'}{2}=\frac{a+b+c}{4}=\frac{p}{2}; \tag{3.4}$$

and, from the equalities (3.2) and (3.4), it follows that:

$$r'=\frac{S'}{p'}=\frac{S}{2\cdot p}=\frac{r}{2} \tag{3.5}$$

and

$$R'=\frac{a'\cdot b'\cdot c'}{4\cdot S'}=\frac{a\cdot b\cdot c}{8\cdot S}=\frac{R}{2}. \tag{3.6}$$

3) Equalities (2.22), (2.23) and (2.24) remain valid. Moreover, segments [A′B′], [B′C′] and [C′A′] are midlines in triangle ABC. So,

$A'B' \parallel AB,$ $B'C' \parallel BC,$ $C'A' \parallel AC$

and:

$\sphericalangle C'A'B=\sphericalangle ACB=C,$ $\sphericalangle B'A'C=\sphericalangle ABC=B.$

It follows that:

$\sphericalangle C'A'B'=\pi$-B-C=A, $\sphericalangle A'B'C'=\pi$-A-C=B

and $\sphericalangle B'C'A'=\pi$-A-B=C;

so:

$\sphericalangle A'=A$, $\sphericalangle B'=B$ and $\sphericalangle C'=C$. (3.7)

On the other hand,

$\sphericalangle B_1A_1C_1 \overset{not.}{=} A_1=\sphericalangle BGC-A$, $\sphericalangle A_1B_1C_1 \overset{not.}{=} B_1=\sphericalangle AGC-B$,

and $\sphericalangle A_1C_1B_1 \overset{not.}{=} C_1=\sphericalangle AGB-C$. (3.8)

But, we know that:

$AA'^2 \overset{not.}{=} m_a^2=\frac{2\cdot(b^2+c^2)-a^2}{4}$, $BB'^2 \overset{not.}{=} m_b^2=\frac{2\cdot(a^2+c^2)-b^2}{4}$

and $CC'^2 \overset{not.}{=} m_c^2=\frac{2\cdot(a^2+b^2)-c^2}{4}$. (3.9)

Applying the Cosine Theorem to the triangle BGC, we obtain that:

$BC^2=BG^2+CG^2-2\cdot BG\cdot GC\cdot\cos(\sphericalangle BGC)$,

namely:

$$a^2=\frac{4}{9}(m_b^2+m_c^2)-2\cdot\frac{2}{3}\cdot m_b\cdot\frac{2}{3}\cdot m_c\cdot\cos(\sphericalangle BGC);$$

from which it follows that:

$$\cos(\sphericalangle BGC)=\frac{\frac{4}{9}\cdot(m_b^2+m_c^2)-a^2}{\frac{8}{9}\cdot m_b\cdot m_c}. \quad (3.10)$$

From the equalities (3.9) and (3.10), it follows that:

$$8\cdot m_b\cdot m_c\cdot\cos(\sphericalangle BGC)=b^2+c^2-5a^2, \text{ namely: } \cos(\sphericalangle BGC)=\frac{b^2+c^2-5\cdot a^2}{8\cdot m_b\cdot m_c}. \quad (3.11)$$

Then,

$$S=S_{\Delta ABC}=3\cdot S_{\Delta BGC}=3\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot m_b\cdot\frac{2}{3}\cdot m_c\cdot\sin(\sphericalangle BGC);$$

from which it follows that:

$$\sin(\sphericalangle BGC)=\frac{3\cdot S}{2\cdot m_b\cdot m_c}. \quad (3.12)$$

From the equalities (3.11) and (3.12), it follows that:

$$tg(\sphericalangle BGC)=\frac{12\cdot S}{b^2+c^2-5\cdot a^2}. \quad (3.13)$$

Analogously, we obtain that:

$$\mathrm{tg}(\measuredangle AGC)=\frac{12\cdot S}{a^2+c^2-5\cdot b^2} \quad \text{and} \quad \mathrm{tg}(\measuredangle AGB)=\frac{12\cdot S}{a^2+b^2-5\cdot c^2}. \qquad (3.13')$$

Now, according to the equalities (3.8), (3.13) and (3.13′), it follows that:

$$A_1=\mathrm{arctg}\left(\frac{12\cdot S}{b^2+c^2-5\cdot a^2}\right)-A, \qquad B_1=\mathrm{arctg}\left(\frac{12\cdot S}{a^2+c^2-5\cdot b^2}\right)-B,$$

and

$$C_1=\mathrm{arctg}\left(\frac{12\cdot S}{a^2+b^2-5\cdot c^2}\right)-C. \qquad (3.14)$$

Now:

4) The equalities (2.25′) become:

$$a_1=2\cdot R\cdot \sin A_1=2\cdot R\cdot \sin\left[\mathrm{arctg}\left(\frac{12\cdot S}{b^2+c^2-5\cdot a^2}\right)-A\right]$$ and the

analogues:

$$b_1=2\cdot R\cdot \sin B_1=2\cdot R\cdot \sin\left[\mathrm{arctg}\left(\frac{12\cdot S}{a^2+c^2-5\cdot b^2}\right)-B\right],$$

$$c_1=2\cdot R\cdot \sin C_1=2\cdot R\cdot \sin\left[\mathrm{arctg}\left(\frac{12\cdot S}{a^2+b^2-5\cdot c^2}\right)-B\right]. \qquad (3.15)$$

5) The equalities (2.26) become:

$$p_1=\frac{a_1+b_1+c_1}{2}=\frac{2\cdot R_1\cdot \sin A_1+2\cdot R_1\cdot \sin B_1+2\cdot R_1\cdot \sin C_1}{2}$$

$$=R\cdot(\sin A_1+\sin B_1+\sin C_1)=4\cdot R\cdot \cos\frac{A_1}{2}\cdot \cos\frac{B_1}{2}\cdot \cos\frac{C_1}{2}$$

$$=4\cdot R\cdot \cos\frac{\alpha-A}{2}\cdot \cos\frac{\beta-B}{2}\cdot \cos\frac{\gamma-C}{2}, \qquad (3.16)$$

where:

$$\alpha=\mathrm{arctg}\left(\frac{12\cdot S}{b^2+c^2-5\cdot a^2}\right), \qquad \beta=\mathrm{arctg}\left(\frac{12\cdot S}{a^2+c^2-5\cdot b^2}\right),$$

and

$$\gamma=\mathrm{arctg}\left(\frac{12\cdot S}{a^2+b^2-5\cdot c^2}\right), \qquad (3.17)$$

in which case:

$$A_1=\alpha-A, \qquad B_1=\beta-B \qquad \text{and} \qquad C_1=\gamma-C. \qquad (3.18)$$

6) The equalities (2.27) become

$$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot \sin A_1\cdot \sin B_1\cdot \sin C_1=2\cdot R^2\cdot \sin(\alpha-A)\cdot \sin(\beta-B)\cdot \sin(\gamma-C). \qquad (3.19)$$

7) The equalities (2.28) become:

$$r_1=4\cdot R_1\cdot \sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=4\cdot R\cdot\sin\frac{\alpha-A}{2}\cdot\sin\frac{\beta-B}{2}\cdot\sin\frac{\gamma-C}{2},\quad(3.20)$$

or, from the equalities (3.16) and (3.19):

$$r_1=\frac{S_1}{p_1}=\frac{2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1}{R\cdot\cos\frac{A_1}{2}\cdot\cos\frac{B_1}{2}\cdot\cos\frac{B_1}{2}}=4\cdot R\cdot\sin\frac{\alpha-A}{2}\cdot\sin\frac{\beta-B}{2}\cdot\sin\frac{\gamma-C}{2}.$$

Now the lengths AA', BB' și CC' become the lengths of the medians of triangle ABC.

8) From the equalities (2.29), (2.29′) and (2.29″) we obtain the equalities:

$$m_a^2=\frac{2\cdot(b^2+c^2)-a^2}{4},\qquad m_b^2=\frac{2\cdot(a^2+c^2)-b^2}{4},$$

and

$$m_c^2=\frac{2\cdot(a^2+b^2)-c^2}{4}.\qquad(3.21)$$

9) The equalities (2.34), (2.34′) and (2.34″) become:

$$A'A_1=\frac{a^2}{2\cdot\sqrt{2\cdot(b^2+c^2)-a^2}},\qquad B'B_1=\frac{b^2}{2\cdot\sqrt{2\cdot(a^2+c^2)-b^2}}$$

and

$$C'C_1=\frac{c^2}{2\cdot\sqrt{2\cdot(a^2+b^2)-c^2}}.\qquad(3.22)$$

10) The equalities (2.36) and (2.36′) become:

$$\frac{GA'}{GA}=\frac{GB'}{GB}=\frac{GC'}{GC}=\frac{1}{2},\qquad(3.23)$$

in which case the inequality from (2.37) becomes equality:

$$\frac{GA'}{GA}\cdot\frac{GB'}{GB}\cdot\frac{GC'}{GC}=\frac{1}{8}.\qquad(3.24)$$

11) The equalities (2.42) and (2.42′) become:

$$S_{\Delta BA_1C}=\frac{a^2}{2\cdot(b^2+c^2)-a^2}\cdot S,\qquad S_{\Delta CB_1A}=\frac{b^2}{2\cdot(a^2+c^2)-b^2}\cdot S$$

and

$$S_{\Delta AC_1B}=\frac{c^2}{2\cdot(a^2+b^2)-c^2}\cdot S,\qquad(3.25)$$

in which case the inequality (2.43) becomes an obvious equality.

12) The relationships (2.45) and (2.45′) become the equalities from (3.22).

13) The equalities (2.46), (2.46′) and (2.46″) become:

$$AA_1=\frac{b^2+c^2}{\sqrt{2\cdot(b^2+c^2)-a^2}};\qquad BB_1=\frac{a^2+c^2}{\sqrt{2\cdot(a^2+c^2)-b^2}};$$

and $$CC_1=\frac{a^2+b^2}{\sqrt{2\cdot(a^2+b^2)-c^2}}. \qquad (3.26)$$

The following remarks is required here:

1. The equalities (3.3), (3.3′) and (3.3″) can also be obtained by applying the Cosine Theorem in triangle AB′C′ and taking into account the same theorem in triangle ABC. □

2. It can be shown, quite easily, that in any triangle ABC the double inequality holds - see (Andrica, Jecan & Magdaș, 2019, p. 223):

$$\frac{b+c}{2}\le AA'\le\frac{b+c}{2}\cdot\cos\frac{A}{2}$$ and the analogues:

$$\frac{a+c}{2}\le BB'\le\frac{a+c}{2}\cdot\cos\frac{B}{2},\qquad \frac{a+b}{2}\le CC'\le\frac{a+b}{2}\cdot\cos\frac{C}{2}. \qquad (3.27)$$

From the equalities (3.27) and according to (Andrica, Jecan & Magdaș, 2019, p. 319), it follows that, in any triangle ABC, the double inequality holds:

$$2\cdot p\le AA'+BB'+CC'\le 4\cdot R+r. \qquad (3.28)$$

3. From equalities (3.22) and according to (Andrica, Jecan & Magdaș, 2019, p. 320), it follows that:

$$\frac{A'A_1}{a}+\frac{B'B_1}{b}+\frac{C'C_1}{c}>\frac{3}{4}. \qquad (3.29)$$

4. Because, according to (Chirciu, 2019, p. 57),

$$GA+GB+GC\ge GA_1+GB_1+GC_1,$$

it follows that:

$$AA'+BB'+CC'\ge 3\cdot(A'A_1+B'B_1+C'C_1). \qquad (3.30)$$

5. As it follows from the equalities (3.15), the expressions of a_1, b_1 and c_1 are not very simple. However, it can be shown, not very easily, that (see Chirciu, 2019, p. 151):

$$a_1^2+b_1^2+c_1^2=\frac{4}{3}\cdot(AA'^2+BB'^2+CC'^2)=a^2+b^2+c^2. \qquad (3.31)$$

6. Also, the expression of S_1, from (3.19), is not very beautiful either. However, we will show that:

$S_1\ge S$.

First, according to equalities (3.22) and (3.21), we observe that:

$$GA_1=GA'+A'A_1=\frac{1}{3}\cdot m_a+\frac{a^2}{4\cdot m_a}=\frac{a^2+b^2+c^2}{6\cdot m_a}=\frac{2}{9}\cdot\frac{m_a^2+m_b^2+m_c^2}{m_a}. \qquad (3.32)$$

Analogously we obtain that:

$$GB_1=\frac{2}{9}\cdot\frac{m_a^2+m_b^2+m_c^2}{m_b}\quad\text{and}\quad GC_1=\frac{2}{9}\cdot\frac{m_a^2+m_b^2+m_c^2}{m_c}. \qquad (3.32')$$

We also note that:

$$\frac{S_{\Delta GA_1B_1}}{S_{\Delta GAB}}=\frac{GA_1\cdot GB_1}{GA\cdot GB}=\frac{1}{9}\cdot\frac{(m_a^2+m_b^2+m_c^2)^2}{m_a^2\cdot m_b^2}. \qquad (3.33)$$

Because,

$$S_{\Delta GAB}=\frac{1}{3}\cdot S, \qquad (3.34)$$

from the equality (3.33) and the inequality (3.34), it follows that:

$$S_{\Delta GA_1B_1}=\frac{1}{27}\cdot\frac{(m_a^2+m_b^2+m_c^2)^2}{m_a^2\cdot m_b^2}\cdot S. \qquad (3.35)$$

Therefore,

$$S_1=S_{\Delta GA_1B_1}+S_{\Delta GA_1C_1}+S_{\Delta GB_1C_1}=\frac{1}{27}\cdot\frac{(m_a^2+m_b^2+m_c^2)^3}{m_a^2\cdot m_b^2\cdot m_c^2}\cdot S\geq S,$$

because, due to the inequality of the means:

$$\frac{1}{27}\cdot\frac{(m_a^2+m_b^2+m_c^2)^3}{m_a^2\cdot m_b^2\cdot m_c^2}\geq 1.$$

7. Of course, using the above results, other interesting results regarding S′ can be obtained; for example (see Chirciu, 2019, p. 61):

$$9\cdot AG\cdot GA_1=9\cdot BG\cdot GB_1=9\cdot CG\cdot GC_1=a^2+b^2+c^2. \qquad (3.36)$$

Hint: We use the equalities (3.22) and the fact that:

$$AG=\frac{2}{3}\cdot m_a \quad \text{and} \quad GA'=\frac{1}{3}\cdot m_a,$$

in which case:

$$9\cdot AG\cdot GA_1=9\cdot\frac{2}{3}\cdot m_a\cdot(GA'+A'A_1)=2\cdot m_a^2+\frac{3}{2}\cdot m_a^2.$$

8. Now, summing the equalities from (3.36) and taking into account the fact that:

$$a^2+b^2+c^2=2\cdot(p^2-r^2-4\cdot R\cdot r), \qquad (3.37)$$

from equalities (3.36) and (3.37), we obtain that:

$$AG\cdot GA_1+BG\cdot GB_1+CG\cdot GC_1=\frac{2}{3}\cdot(p^2-r^2-4\cdot R\cdot r). \qquad (3.38)$$

9. We can obtain very interesting inequalities. For example, starting from the fact that in any acute triangle ABC, (see Andrica, Jecan & Magdaș, 2019, p. 143):

$$a^2+b^2+c^2\geq 4\cdot(R+r)^2, \qquad (3.39)$$

we obtain the following inequality,

$$AG\cdot GA_1+BG\cdot GB_1+CG\cdot GC_1\geq 4\cdot(R+r)^2, \qquad (3.40)$$

or, in other words:

$$p^2-r^2-4\cdot R\cdot r\geq 6\cdot(R+r)^2. \qquad (3.41)$$

10. Also from the equality (3.38), we obtain that:

$$4\cdot r\cdot(2\cdot R-r)\leq AG\cdot GA_1+BG\cdot GB_1+CG\cdot GC_1\leq \frac{4}{3}\cdot(2\cdot R^2+r^2). \qquad (3.42)$$

Hint: It is shown that:

$$p^2-r^2-4\cdot R\cdot r\in[6\cdot r\cdot(2\cdot R-r),2\cdot(2\cdot R^2+r^2)].$$

11. If:

- $A''B''C''$ is the G'-cevian triangle of the triangle $A'B'C'$, that is, the median triangle of this triangle $A'B'C'$;
- $A'''B'''C'''$ is the G''-cevian triangle of the triangle $A''B''C''$, that is, the median triangle of this triangle $A''B''C''$;
- $A^{(iv)}B^{(iv)}C^{(iv)}$ is the G'''-cevian triangle of the triangle $A'''B'''C'''$, that is, the median triangle of this triangle $A'''B'''C'''$;
- ...
- $A^{(n)}B^{(n)}C^{(n)}$ is the $G^{(n-1)}$-cevian triangle of the triangle $A^{(n-1)}B^{(n-1)}C^{(n-1)}$, that is, the median triangle of this triangle $A^{(n-1)}B^{(n-1)}C^{(n-1)}$,

then the following equalities hold:

$A'=A''=A'''=A^{(iv)}=...=A^{(n)}=A$, $\quad B'=B''=B'''=B^{(iv)}=...=B^{(n)}=B$,

and $\quad C'=C''=C'''=C^{(iv)}=...=C^{(n)}=C$; $\qquad$ (3.43)

$$a'=\frac{a}{2},\quad a''=\frac{a'}{2}=\frac{a}{4},\quad a'''=\frac{a''}{2}=\frac{a'}{4}=\frac{a}{8},\quad a^{(iv)}=\frac{a'''}{2}=\frac{a''}{4}=\frac{a'}{8}=\frac{a}{16},$$

$$\ldots\quad a^{(n)}=\frac{a^{(n-1)}}{2}=\frac{a^{(n-2)}}{2^2}=...=\frac{a''}{2^{n-2}}=\frac{a'}{2^{n-1}}=\frac{a}{2^n};\ (3.44)$$

$$b'=\frac{b}{2},\quad b''=\frac{b'}{2}=\frac{b}{4},\quad b'''=\frac{b''}{2}=\frac{b'}{4}=\frac{b}{8},\quad b^{(iv)}=\frac{b'''}{2}=\frac{b''}{4}=\frac{b'}{8}=\frac{b}{16},$$

$$\ldots\quad b^{(n)}=\frac{b^{(n-1)}}{2}=\frac{b^{(n-2)}}{2^2}=...=\frac{b''}{2^{n-2}}=\frac{b'}{2^{n-1}}=\frac{b}{2^n};\qquad (3.44')$$

$$c'=\frac{c}{2},\quad c''=\frac{c'}{2}=\frac{c}{4},\quad c'''=\frac{c''}{2}=\frac{c'}{4}=\frac{c}{8},\quad c^{(iv)}=\frac{c'''}{2}=\frac{c''}{4}=\frac{c'}{8}=\frac{c}{16},$$

$$\ldots\quad c^{(n)}=\frac{c^{(n-1)}}{2}=\frac{c^{(n-2)}}{2^2}=...=\frac{c''}{2^{n-2}}=\frac{c'}{2^{n-1}}=\frac{c}{2^n};\ (3.44'')$$

$$p'=\frac{p}{2},\quad p''=\frac{p'}{2}=\frac{p}{4},\quad p'''=\frac{p''}{2}=\frac{p'}{4}=\frac{p}{8},\quad p^{(iv)}=\frac{p'''}{2}=\frac{p''}{4}=\frac{p'}{8}=\frac{p}{16},$$

$$\ldots\quad p^{(n)}=\frac{p^{(n-1)}}{2}=\frac{p^{(n-2)}}{2^2}=...=\frac{p''}{2^{n-2}}=\frac{p'}{2^{n-1}}=\frac{p}{2^n};\qquad (3.45)$$

$$R'=\frac{R}{2},\ R''=\frac{R'}{2}=\frac{R}{4},\quad R'''=\frac{R''}{2}=\frac{R'}{4}=\frac{R}{8},\ R^{(iv)}=\frac{R'''}{2}=\frac{R''}{4}=\frac{R'}{8}=\frac{R}{16},$$

$$\ldots\quad R^{(n)}=\frac{R^{(n-1)}}{2}=\frac{R^{(n-2)}}{2^2}=\ldots=\frac{R''}{2^{n-2}}=\frac{R'}{2^{n-1}}=\frac{R}{2^n}; \tag{3.46}$$

$$r'=\frac{r}{2},\ r''=\frac{r'}{2}=\frac{r}{4},\quad r'''=\frac{r''}{2}=\frac{r'}{4}=\frac{r}{8},\quad r^{(iv)}=\frac{r'''}{2}=\frac{r''}{4}=\frac{r'}{8}=\frac{r}{16},$$

$$\ldots\quad r^{(n)}=\frac{r^{(n-1)}}{2}=\frac{r^{(n-2)}}{2^2}=\ldots=\frac{r''}{2^{n-2}}=\frac{r'}{2^{n-1}}=\frac{r}{2^n}; \tag{3.47}$$

$$S'=\frac{S}{4},\ S''=\frac{S'}{4}=\frac{S}{16},\quad S'''=\frac{S''}{4}=\frac{S'}{16}=\frac{S}{64},\quad S^{(iv)}=\frac{S'''}{4}=\frac{S''}{16}=\frac{S'}{64}=\frac{S}{256},$$

$$\ldots\quad S^{(n)}=\frac{S^{(n-1)}}{4}=\frac{S^{(n-2)}}{4^2}=\ldots=\frac{S''}{4^{n-2}}=\frac{S'}{4^{n-1}}=\frac{S}{4^n}. \tag{3.48}$$

We conclude this paragraph with the following six clarifications:

12. According to the equalities (3.46) and (3.47), Euler's inequality

$$R\geq 2\cdot r, \tag{3.49}$$

we can look at it as occurring, for every $n\in\mathbf{N^*}$, in any triangle $A^{(n)}B^{(n)}C^{(n)}$.

13. For every $n\in\mathbf{N^*}$, all triangles $A^{(n)}B^{(n)}C^{(n)}$ have the same centroid.

14. For any point M in the plane, the equality holds:

$$\overrightarrow{MA'}+\overrightarrow{MB'}+\overrightarrow{MC'}=\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}. \tag{3.50}$$

Hint: The equalities obtained from the Median Theorem are used in vector form:

$$\overrightarrow{MA'}=\frac{1}{2}\cdot(\overrightarrow{MA}+\overrightarrow{MB}) \quad \text{and the analogues.} \tag{3.51}$$

15. The center of the circumscribed circle of triangle ABC coincides with the orthocenter of the triangle A′B′C′.

16. Triangles ABC and A′B′C′ are homothetic triangles, by homothety $H_{G,-\frac{1}{2}}$, in the sense that:

$$H_{G,-\frac{1}{2}}(\Delta ABC)=\Delta A'B'C'. \tag{3.52}$$

17. Triangles ABC and A″B″C″ are orthologic triangles - see (Pătrașcu & Smarandache, p. 40)

Conclusions and Recommendations

As in the other two papers with the same name (Vălcan, 2021, 2022), we showed that we can study, in the general way, using logical deductibility and the method of analogy, the Cevian and Circumcevian triangles, in this paper we showed how to transpose the results obtained in the general way, in the case of G-cevian triangles (also called the median triangle), respectively G-circumcevian.

About the principles underlying these two methods and their application in the didactic act, see (Vălcan, 2013). Of course, and in this case, as I have already stated, the reader attentive and interested in these matters, using usual mathematical knowledge, valid in any triangle, such as those presented in (Andrica, Jecan & Magdaș, 2019), (Chirciu, 2014, 2015, (I) 2021, (II) 2021), (Coța et al 1982) and / or (Țigănilă & Dumitru, 1979), can obtain a series of other very interesting geometric or trigonometric identities and inequalities, some of them very difficult to prove, synthetically. On the other hand, all these geometric or trigonometric relations introduced in certain derivable or only integrable functions can lead to a series of differential or integral identities or inequalities, particularly interesting, such as those presented in (Stănescu, 2015).

As I stated at the beginning, the work is, exclusively, of the Didactics of Mathematics and is addressed, equally, to pupils, students and teachers eager for performance, in this field of Mathematics or, in Mathematics, in general.

References:

Andrica, D., Jecan, E., & Magdaș, C. M. (2019). *Geometrie – teme și probleme pentru grupele de excelență (Geometry - topics and issues for excellence groups)*. Editura Paralela 45

Brânzei, D. & Zanoschi, A. (1999). *Geometrie – probleme cu vectori (Geometry - problems with vectors)*. Editura Paralela 45

Chirciu, M. (2014). *Inegalități algebrice - de la inițiere la performanță (Algebraic inequalities - from initiation to performance)*. Vol. I, Editura Paralela 45.

Chirciu, M. (2015). *Inegalități geometrice - de la inițiere la performanță (Geometric inequalities - from initiation to performance)*. Vol. I, Editura Paralela 45.

Chirciu, M. (2019). *Puncte remarcabile într-un triunghi (Remarkable points in a triangle)*. Editura Paralela 45

Coța, A. & colectiv (1982). *Matematică, Manual pentru clasa a X – a. Geometrie și trigonometrie (Mathematics, Textbook for the 10th grade. Geometry and Trigonometry)*. Editura didactică și pedagogică

Stănescu, F. (2015). *Inegalități integrale (Integral inequalities)*. Editura Paralela 45

Pătrașcu, I. & Smarandache, F. (2020). *Geometria triunghiurilor ortologice* (*Geometry of orthologic triangles*). Editura Agora

Țigănilă, Gh. & Dumitru, M. I. (1979). *Culegere de probleme de matematici (Collection of math problems)*, Editura Scrisul românesc

Vălcan, D. (2013). *Didactica Matematicii (Didactics of Mathematics)*. Editura Matrix Rom.

Vălcan, D. (2021). *Deductibility and analogy in the study of some triangles (I) - general results*-, in „*Education, society, family interdisciplinary perspectives and analyses*”. Editura EIKON, pp. 61-71

Vălcan, D. (2022). *Deductibility and analogy in the study of some triangles (II) – the I-cevian triangle and the I-circumcevian triangle* -, în „*Values, models, education. Contemporary perspectives*”. Editura EIKON, pp 89-106