AGAIN ABOUT TRIGONOMETRIC FUNCTIONS AND HYPERBOLIC FUNCTIONS OF COMPLEX ARGUMENT

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Abstract. After defining hyperbolic functions in five previous papers and presenting and proving their 184 properties, in a recent paper we extended these functions from the set of real numbers - **R** to the set of complex numbers - **C**. Thus, we defined the set C of complex numbers, all known hyperbolic functions: sh - hyperbolic sine, ch - hyperbolic cosine, th - hyperbolic tangent, cth - hyperbolic cotangent, sch - hyperbolic secant and csh - hyperbolic cosecant, also the inverses of these functions. But, to do this, we extended from **R** to **C** the known functions: ex - exponential, lnx - logarithmic, but also the trigonometric ones: sin - sine, cos - cosine, tg - tangent, ctg - cotangent, sec - secant and cosec - cosecant, as well as their inverses. We then presented 18 immediate properties of these functions, properties divided into two groups: A) The periodicity of hyperbolic functions of complex argument and B) The values of hyperbolic functions and trigonometric functions of imaginary argument. In this paper we will continue with the presentation of 36 other properties of these functions, also divided into two groups: C) The values of the hyperbolic functions of complex argument, their conjugates and modules and D) The values of the trigonometric functions of complex argument, their conjugates and modules.

Keywords: function, sine, cosine, tangent, cotangent, trigonometric, hyperbolic, complex argument

1. Putting the problem

As mentioned above, in five previous papers I have defined hyperbolic functions and presented and proved 184 of their properties - see my papers in the bibliography.

In a recent paper (Vălcan, 2022) we extended these functions from the set of real numbers - **R** to the set of complex numbers - **C**, or, in other words, from the real line to the complex plane. Thus, we defined on the set **C** of complex numbers, all known hyperbolic functions: sh - hyperbolic sine, ch - hyperbolic cosine, tg - hyperbolic tangent, cth - hyperbolic cotangent, sch - hyperbolic secant and csh - hyperbolic cosecant, also the inverse these functions. But in order to do this, we had to extend from **R** to **C** the known functions: ex - exponential, lnx - logarithmic, but also the trigonometric ones: sin - sine, cos - cosine, tg - tangent, ctg - cotangent, sec

- secant and cosec - cosecant, as well as their inverses. Then, we presented and proved 18 immediate properties of these functions, which we divided into two groups:

A) *The periodicity of hyperbolic functions of complex argument* and

B) *The values of hyperbolic functions and trigonometric functions of imaginary argument*.

In this paper we will continue with the presentation of another 36 properties of these functions, properties also divided into two groups:

C) *The values of the hyperbolic functions of complex argument, their conjugates and modules* and

D) *The values of the trigonometric functions of complex argument, their conjugates and modules*.

Because, as I just mentioned, this work is a continuation of the work (Vălcan, 2022), in the first part, we have resumed the definitions and main technical results from (Vălcan, 2022), which we need in the presentation and proof of these 36 of new properties. Thus, we can say that we have defined, presented and proved 54 properties of these functions, showing that everything we have defined, presented and proved in the works mentioned above, for real argument functions, can be extended to the same complex argument functions.

Of course, like the paper (Vălcan, 2022), this paper, through its content, is closer to what we call the University Didactics of Mathematics, than to what is the Didactics of Mathematics for pre-university education, although in high school excellence groups may these issues are presented. We say this because in all the treatises on Complex Analysis these functions are defined and used in large-scale scientific developments.

2. Presentation of technical results

2. As mentioned above, in this paragraph we present the definitions and main technical results presented and proven in (Vălcan, 2022), and which we need in the developments in the next paragraph. We keep notations and numbering from there.

Definition 2.5: *We define the functions:*

$$sin: \mathbf{C} \to \mathbf{C} \qquad \text{and} \qquad cos: \mathbf{C} \to \mathbf{C},$$

thus: for every $z \in \mathbf{C}$,

$$sinz = \frac{e^{i\cdot z} - e^{-i\cdot z}}{2\cdot i} \qquad \text{and} \qquad cosx = \frac{e^{i\cdot z} + e^{-i\cdot z}}{2}. \qquad (2.19)$$

Definitions 2.7: *For every* $z \in \mathbf{C}$, *for which* $cosz \neq 0$, *we define*:

$$tgz = \frac{\sin z}{\cos z} \qquad \text{and} \qquad secz = \frac{1}{\cos z} \qquad (2.22)$$

and, for every $z \in \mathbf{C}$*, for which* $\sin z \neq 0$*, we define*:

$$ctgz = \frac{\cos z}{\sin z} \qquad \text{and} \qquad cosecz = \frac{1}{\sin z}. \quad (2.23)$$

Definition 2.8: *The function* $sh : \mathbf{C} \rightarrow \mathbf{C}$*, given by law, for every* $z \in \mathbf{C}$*,*

$$sh(z) = \frac{e^{z} - e^{-z}}{2} \overset{not.}{=} shz, \qquad (2.24)$$

is called ***the hyperbolic sine function*** *of argument z.*

Definition 2.9: *The function* $ch : \mathbf{C} \rightarrow \mathbf{C}$*, given by law, for every* $z \in \mathbf{C}$*,*

$$ch(z) = \frac{e^{z} + e^{-z}}{2} \overset{not.}{=} chz, \qquad (2.25)$$

is called ***the hyperbolic cosine function*** *of argument z.*

Definition 2.11: *For every* $z \in \mathbf{C}$*, for which* $chz \neq 0$*, we define the function,*

$$th(z) = \frac{sh(z)}{ch(z)} \overset{not.}{=} thz, \qquad (2.29)$$

and which is called ***the hyperbolic tangent function*** *of argument z.*

Definition 2.12: *For every* $z \in \mathbf{C}$*, for which* $shz \neq 0$*, we define,*

$$cth(z) = \frac{ch(z)}{sh(z)} \overset{not.}{=} cthz, \qquad (2.30)$$

and which is called ***the hyperbolic sine function*** *of argument z.*

Definition 2.13: *For every* $z \in \mathbf{C}$*, for which* $chz \neq 0$*, we define,*

$$sch(z) = \frac{1}{ch(z)} \overset{not.}{=} schz, \qquad (2.31)$$

and which is called ***the hyperbolic secant function*** *of argument z.*

Definition 2.14: *For every* $z \in \mathbf{C}$*, for which* $shz \neq 0$*, we define,*

$$csh(z) = \frac{1}{sh(z)} \overset{not.}{=} cshz, \qquad (2.32)$$

and which is called ***the hyperbolic cosecant function*** *of argument z.*

Then we defined the following inverse trigonometric functions:

$\sin^{-1}z=\frac{1}{i}\cdot\ln(i\cdot z+\sqrt{1-z^2})$. (2.33) $\cos^{-1}z=\frac{1}{i}\cdot\ln(z+\sqrt{z^2-1})$, (2.34)

$tg^{-1}z=\frac{1}{2\cdot i}\cdot\ln\left(\frac{1+i\cdot z}{1-i\cdot z}\right)$, (2.35) $ctg^{-1}z=\frac{1}{2\cdot i}\cdot\ln\left(\frac{z+i}{z-i}\right)$, (2.36)

$\sec^{-1}z=\frac{1}{i}\cdot\ln\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}-1}\right)$, (2.37) $cosec^{-1}z=\frac{1}{i}\cdot\ln\left(\frac{i}{z}+\sqrt{1-\frac{1}{z^2}}\right)$, (2.38)

$sh^{-1}z=\ln(z+\sqrt{z^2+1})$, (2.39) $ch^{-1}z=\ln(z+\sqrt{z^2-1})$, (2.40)

$th^{-1}z=\frac{1}{2}\cdot\ln\left(\frac{1+z}{1-z}\right)$, (2.41) $cth^{-1}z=\frac{1}{2}\cdot\ln\left(\frac{z+1}{z-1}\right)$, (2.42)

$sch^{-1}z=\ln\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}-1}\right)$, (2.43) $csh^{-1}z=\ln\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}+1}\right)$. (2.44)

From the above equations, we immediately obtain that (and) the following equalities take place:

Remark.15: *The following statements hold*:

1) *For every* $x\in\mathbf{R}$,

$\sin^{-1}(ix)=i\cdot sh^{-1}x$; *or, otherwise*: $sh^{-1}x=-i\cdot\sin^{-1}(ix)$; (5.29)

$sh^{-1}(ix)=i\cdot\sin^{-1}x$; *or, otherwise*: $\sin^{-1}x=-i\cdot sh^{-1}(ix)$; (5.30)

$\cos^{-1}x=\pm i\cdot ch^{-1}x$; *or, otherwise*: $ch^{-1}x=\pm i\cdot\cos^{-1}x$; (5.31)

$tg^{-1}(ix)=i\cdot th^{-1}x$; *or, otherwise*: $th^{-1}x=-i\cdot tg^{-1}(ix)$; (5.32)

$th^{-1}(ix)=i\cdot tg^{-1}x$; *or, otherwise*: $tg^{-1}x=-i\cdot th^{-1}(ix)$; (5.33)

2) *For every* $x\in\mathbf{R}^*$,

$ctg^{-1}(ix)=-i\cdot cth^{-1}x$; *or, otherwise*: $cth^{-1}x=i\cdot ctg^{-1}(ix)$; (5.34)

$cth^{-1}(ix)=-i\cdot ctg^{-1}x$; *or, otherwise*: $ctg^{-1}x=i\cdot cth^{-1}(ix)$; (5.35)

$cosec^{-1}(ix)=-i\cdot csh^{-1}x$; *or, otherwise*: $csh^{-1}x=i\cdot cosec^{-1}(ix)$; (5.36)

$csh^{-1}(ix)=-i\cdot cosec^{-1}x$; *or, otherwise*: $cosec^{-1}x=i\cdot csh^{-1}(ix)$; (5.37)

3) *For every* $x \in \mathbf{R}$,

$$sec^{-1}x = \pm i \cdot sch^{-1}x; \qquad \text{or, otherwise:} \quad sch^{-1}x = \pm i \cdot sec^{-1}x; \qquad (5.38)$$

$$sch^{-1}x = \pm i \cdot sec^{-1}x; \qquad \text{or, otherwise:} \quad sec^{-1}x = \pm i \cdot sch^{-1}x. \qquad (5.39')$$

We leave the proof of the equalities (2.34) - (2.44) to the reader who is attentive and interested in these issues.

3. Immediate properties of hyperbolic functions of complex argument

We present and prove, below, a list of 36 immediate properties of hyperbolic functions of complex argument, divided into two groups:

C) *The values of the hyperbolic functions of complex argument, their conjugates and modules* and

D) *The values of the trigonometric functions of complex argument, their conjugates and modules*

As these results are a continuation of those in the paper (Vălcan, 2022), the numbering will be a continuation of those there.

Thus, we have the following results:

Proposition 3.1: *The following statements hold*:

C) *The values of the hyperbolic functions of complex argument, their conjugates and modules*

19) *For every* $x, y \in \mathbf{R}$,

$$sh(x+iy) = shx \cdot cosy + i \cdot chx \cdot siny. \qquad (3.18)$$

20) *For every* $x, y \in \mathbf{R}$,

$$sh(x-iy) = shx \cdot cosy - i \cdot chx \cdot siny. \qquad (3.19)$$

or, equivalent, for every $z = x + i \cdot y$:

$$sh\,\overline{z} = \overline{shz}\,. \qquad (3.19')$$

21) *For every* $z = x + i \cdot y \in \mathbf{C}$,

$$|shz|^2 = sh^2x + sin^2y = |sh\,\overline{z}\,|^2. \qquad (3.20)$$

22) *For every* $x, y \in \mathbf{R}$,

$$ch(x+iy) = chx \cdot cosy + i \cdot shx \cdot siny. \qquad (3.21)$$

23) *For every* $x, y \in \mathbf{R}$,

$$ch(x\text{-}iy)=chx\cdot cosy\text{-}i\cdot shx\cdot siny. \quad (3.22)$$

or, equivalent, for every $z=x+i\cdot y$:

$$ch\,\bar{z} = \overline{chz}\,. \quad (3.22')$$

24) *For every* $z=x+i\cdot y\in\mathbf{C}$,

$$|chz|^2=ch^2x+sin^2y=|sh\,\bar{z}\,|^2. \quad (3.23)$$

25) *For every* $x\in\mathbf{R}^*$ *and for every* $y\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$,

$$th(x+iy)=\frac{thx+i\cdot tgy}{1+i\cdot thx\cdot tgy}. \quad (3.24)$$

26) *For every* $x\in\mathbf{R}^*$ *and for every* $y\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$,

$$th(x\text{-}iy)=\frac{thx-i\cdot tgy}{1-i\cdot thx\cdot tgy}; \quad (3.25)$$

or, equivalent, for every $z=x+i\cdot y$ *for which thx and tgy exist*:

$$th\,\bar{z} = \overline{thz}\,. \quad (3.25')$$

27) *For everyz=*$x+i\cdot y\in\mathbf{C}$, *with* $x\in\mathbf{R}^*$ *and* $y\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$,

$$|thz|^2=\frac{th^2x+tg^2y}{1+th^2x\cdot tg^2y}=|th\,\bar{z}\,|^2. \quad (3.26)$$

28) *For every* $x\in\mathbf{R}^*$ *and for every* $y\in\mathbf{R}\setminus\{k\cdot\pi\,/\,k\in\mathbf{Z}\}$,

$$cth(x+iy)=\frac{1-i\cdot cthx\cdot ctgy}{cthx-i\cdot ctgy}. \quad (3.27)$$

29) *For every* $x\in\mathbf{R}^*$ *and for every* $y\in\mathbf{R}\setminus\{k\cdot\pi\,/\,k\in\mathbf{Z}\}$,

$$cth(x\text{-}iy)=\frac{1+i\cdot cthx\cdot ctgy}{cthx+i\cdot ctgy}; \quad (3.28)$$

or, equivalent, for every $z=x+i\cdot y$ *for which cthx and ctgy exist*:

$$cth\,\bar{z} = \overline{cthz}\ . \tag{3.28′}$$

30) *For every* $z=x+i\cdot y\in \mathbf{C}$, *with* $x\in \mathbf{R}^*$ *and* $y\in \mathbf{R}\setminus\{k\cdot\pi \,/\, k\in \mathbf{Z}\}$,

$$|cthz|^2 = \frac{cth^2 x\cdot ctg^2 y+1}{cth^2 x+ctg^2 y} = |cth\,\bar{z}\,|^2. \tag{3.29}$$

31) *For every* $x\in \mathbf{R}^*$ *and for every* $y\in \mathbf{R}\setminus\{k\cdot\pi \,/\, k\in \mathbf{Z}\}$,

$$csh(x+iy) = \frac{shx\cdot cos\,y - i\cdot chx\cdot sin\,y}{sh^2 x+sin^2 y} = \frac{sh(x-i\cdot y)}{sh^2 x+sin^2 y}\,; \tag{3.30}$$

or, equivalent, for every $z=x+i\cdot y$, *for which* $shz\neq 0$,

$$cshz = \frac{sh\bar{z}}{|shz|^2}\ . \tag{3.30′}$$

32) *For every* $x\in \mathbf{R}^*$ *and for every* $y\in \mathbf{R}\setminus\{k\cdot\pi \,/\, k\in \mathbf{Z}\}$,

$$csh(x-iy) = \frac{shx\cdot cos\,y + i\cdot chx\cdot sin\,y}{sh^2 x+sin^2 y} = \frac{sh(x+i\cdot y)}{sh^2 x+sin^2 y}\,; \tag{3.31}$$

or, equivalent, for every $z=x+i\cdot y$, *for which* $shz\neq 0$,

$$csh\,\bar{z} = \frac{shz}{|shz|^2} = \overline{cshz}\ . \tag{3.31′}$$

33) *For every* $z=x+i\cdot y\in \mathbf{C}$, *with* $x\in \mathbf{R}^*$ *and* $y\in \mathbf{R}\setminus\{k\cdot\pi \,/\, k\in \mathbf{Z}\}$,

$$|cshz| = \frac{1}{|shz|} = |csh\,\bar{z}\,|. \tag{3.32}$$

34) *For every* $x\in \mathbf{R}^*$ *and for every* $y\in \mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\,\middle|\, k\in Z\right\}$,

$$sch(x+iy) = \frac{chx\cdot cos\,y - i\cdot shx\cdot sin\,y}{ch^2 x+sin^2 y} = \frac{ch(x-i\cdot y)}{ch^2 x+sin^2 y}\,; \tag{3.33}$$

or, equivalent, for every $z=x+i\cdot y$, *for which* $chz\neq 0$,

$$schz = \frac{ch\bar{z}}{|chz|^2}\ . \tag{3.33′}$$

35) *For every* $x\in\mathbf{R}^*$ *and for every* $y\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$,

$$sch(x\text{-}iy)=\frac{chx\cdot\cos y+i\cdot shx\cdot\sin y}{ch^2x+\sin^2 y}=\frac{ch(x+i\cdot y)}{ch^2x+\sin^2 y}; \qquad (3.34)$$

or, equivalent, for every $z=x+i\cdot y$, *for which* $chz\neq 0$,

$$sch\,\bar{z}=\frac{chz}{|chz|^2}=\overline{schz}\,. \qquad (3.34')$$

36) *For every* $z=x+i\cdot y\in\mathbf{C}$, *cu* $x\in\mathbf{R}^*$ *and* $y\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$,

$$|schz|=\frac{1}{|chz|}=|sch\,\bar{z}\,|. \qquad (3.35)$$

D) *The values of the trigonometric functions of complex argument, their conjugates and modules*

37) *For every* $x, y\in\mathbf{R}$,

$$\sin(x+iy)=\sin x\cdot chy+i\cdot\cos x\cdot shy. \qquad (3.36)$$

38) *For every* $x, y\in\mathbf{R}$,

$$\sin(x\text{-}iy)=\sin x\cdot chy\text{-}i\cdot\cos x\cdot shy; \qquad (3.37)$$

or, equivalent, for every $z=x+i\cdot y$:

$$\sin\bar{z}=\overline{\sin z}\,. \qquad (3.37')$$

39) *For every* $z=x+i\cdot y\in\mathbf{C}$,

$$|\sin z|^2=\sin^2x+sh^2y=|\sin\bar{z}\,|^2. \qquad (3.38)$$

40) *For every* $x, y\in\mathbf{R}$,

$$\cos(x+iy)=\cos x\cdot chy+i\cdot\sin x\cdot shy. \qquad (3.39)$$

41) *For every* $x, y\in\mathbf{R}$,

$$\cos(x\text{-}iy)=\cos x\cdot chy\text{-}i\cdot\sin x\cdot shy, \qquad (3.40)$$

or equivalent, for every $z=x+i\cdot y$:

$$\cos \bar{z} = \overline{\cos z}\,. \qquad (3.40')$$

42) *For every* $z=x+i\cdot y\in\boldsymbol{C}$,

$$|\cos z|^2=\cos^2 x+sh^2 y=|\cos \bar{z}\,|^2. \qquad (3.41)$$

43) *For every* $x\in\boldsymbol{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$ *and* $y\in\boldsymbol{R}$,

$$tg(x+iy)=\frac{tgx+i\cdot thy}{1-i\cdot tgx\cdot thy}; \qquad (3.42)$$

44) *For every* $x\in\boldsymbol{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$ *and* $y\in\boldsymbol{R}$,

$$tg(x\text{-}iy)=\frac{tgx-i\cdot thy}{1+i\cdot tgx\cdot thy}; \qquad (3.43)$$

or, equivalent, for every $z=x+i\cdot y$ *for which tgx and thy exist*:

$$tg\,\bar{z} = \overline{tgz}\,. \qquad (3.43')$$

45) *For every* $z=x+i\cdot y\in\boldsymbol{C}$, *with* $x\in\boldsymbol{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$ *and* $y\in\boldsymbol{R}$,

$$|tgz|^2=\frac{tg^2x+th^2y}{1+tg^2x\cdot th^2y}=|gh\,\bar{z}\,|^2. \qquad (3.44)$$

46) *For every* $x\in\boldsymbol{R}\setminus\left\{\frac{k\cdot\pi}{2}\middle|k\in Z\right\}$ *and for every* $y\in\boldsymbol{R}^*$,

$$ctg(x+iy)=\frac{1-i\cdot ctgx\cdot cthy}{ctgx-i\cdot cthy}. \qquad (3.45)$$

$$ctg(x\text{-}iy)=\frac{1+i\cdot ctgx\cdot cthy}{ctgx+i\cdot cthy}; \qquad (3.46)$$

or, equivalent, for every $z=x+i\cdot y$ *for which ctgx and cthy exist*:

$$ctg\,\bar{z} = \overline{ctgz}\,. \qquad (3.46')$$

48) *For every* $x\in\mathbf{R}\setminus\left\{\frac{k\cdot\pi}{2}\middle|k\in Z\right\}$ *and for every* $y\in\mathbf{R}^*$,

$$|ctgz|^2=\frac{ctg^2x\cdot cth^2y+1}{ctg^2x+cth^2y}=|ctg\,\bar{z}\,|^2. \tag{3.47}$$

49) *For every* $x\in\mathbf{R}\setminus\{k\cdot\pi\,/\,k\in\mathbf{Z}\}$ *and for every* $y\in\mathbf{R}^*$,

$$cosec(x+iy)=\frac{\sin x\cdot chy-i\cdot\cos x\cdot shy}{\sin^2x+sh^2y}=\frac{\sin(x-i\cdot y)}{\sin^2x+sh^2y}; \tag{3.48}$$

or, equivalent, for every $z=x+i\cdot y$*, for which* $\sin z\neq 0$,

$$cosecz=\frac{\sin\bar{z}}{|\sin z|^2}. \tag{3.48′}$$

50) *For every* $x\in\mathbf{R}\setminus\{k\cdot\pi\,/\,k\in\mathbf{Z}\}$ *and for every* $y\in\mathbf{R}^*$,

$$cosec(x-iy)=\frac{\sin x\cdot chy+i\cdot\cos x\cdot shy}{\sin^2x+sh^2y}=\frac{\sin(x+i\cdot y)}{\sin^2x+sh^2y}; \tag{3.49}$$

or, equivalent, for every $z=x+i\cdot y$*, for which* $\sin z\neq 0$,

$$cosec\,\bar{z}=\frac{\sin z}{|\sin z|^2}. \tag{3.49′}$$

51) *For every* $x\in\mathbf{R}\setminus\{k\cdot\pi\,/\,k\in\mathbf{Z}\}$ *and for every* $y\in\mathbf{R}^*$,

$$|cosecz|=\frac{1}{|\sin z|}=|cosec\,\bar{z}\,|. \tag{3.50}$$

52) *For every* $x\in\mathbf{R}\setminus\left\{\frac{(2\cdot k+1)\cdot\pi}{2}\middle|k\in Z\right\}$ *and for every* $y\in\mathbf{R}^*$,

$$sec(x+iy)=\frac{\cos x\cdot chy-i\cdot\sin x\cdot shy}{\cos^2x+sh^2y}=\frac{\cos(x-i\cdot y)}{\cos^2x+sh^2y}; \tag{3.51}$$

or, equivalent, for every $z=x+i\cdot y$*, for which* $\cos z\neq 0$,

$$secz=\frac{\cos\bar{z}}{|\cos z|^2}. \tag{3.51′}$$

53) For every $x \in \mathbf{R} \setminus \left\{ \frac{(2\cdot k+1)\cdot\pi}{2} \middle| k\in Z \right\}$ *and for every* $y \in \mathbf{R}^*$,

$$sec(x\text{-}iy) = \frac{cos x\cdot chy + i\cdot sin x\cdot shy}{cos^2 x + sh^2 y} = \frac{cos(x+i\cdot y)}{cos^2 x + sh^2 y}; \qquad (3.52)$$

or, equivalent, for every $z=x+i\cdot y$, *for which* $chz\neq 0$,

$$sec\,\bar{z} = \frac{cos\,z}{|cos\,z|^2} = \overline{sec\,z}\,. \qquad (3.52')$$

54) For every $x \in \mathbf{R} \setminus \left\{ \frac{(2\cdot k+1)\cdot\pi}{2} \middle| k\in Z \right\}$ *and for evey* $y \in \mathbf{R}^*$,

$$|secz| = \frac{1}{|cos\,z|} = |sec\,\bar{z}|. \qquad (3.53)$$

***Proof*: 19)** Equality (3.18) results from the equalities (2.17), (2.18) and (2.24).

20) Equality (3.19) follows from the equalities (2.17) and (2.24).

21) Equality (3.20) follows from and the equality (3.18) and from the equality (3.1) in (Vălcan, 2016).

22) Equality (3.21) follows from the equalities (2.17), (2.18) and (2.25).

23) Equality (3.22) follows from the equalities (2.17) and (2.25).

24) Equality (3.23) follows from and the equality (3.21) and from the equality (3.1) in (Vălcan, 2016).

25) Equality (3.24) follows from the equalities (3.18) and (3.21).

26) Equality (3.25) follows from the equalities (3.19) and (3.22).

27) Equality (3.26) follows from the equalities (3.24) or (3.25).

28) Equality (3.27) follows from the equalities (3.21) and (3.18) or from the equalities (2.30) and (3.24).

29) Equality (3.28) follows from the equalities (3.22) and (3.19) or from the equalities (2.30) and (3.25).

30) Equality (3.29) follows from the equalities (3.27) or (3.28).

31) Equality (3.30) follows from the equalities (2.32) and (3.18).

32) Equality (3.31) follows from the equalities (2.32) and (3.19).

33) Equality (3.32) follows from the equalities (3.30′) and (3.31′).

34) Equality (3.33) follows from the equalities (2.31) and (3.21).

35) Equality (3.34) follows from the equalities (2.31) and (3.22).

36) Equality (3.35) follows from the equalities (3.33′) and (3.34′).

37) Equality (3.36) follows from the equalities (2.19), (3.13′) and (3.14).

38) Equality (3.37) follows from the equalities (2.19), (3.13′) and (3.14), respectively from the equalities (2.9) și (2.10) in (Vălcan, 2016).

39) Equality (3.38) follows from the equalities (3.36) or (3.37).

40) Equality (3.39) follows from the equalities (2.19), (3.13′) and (3.14).

41) Equality (3.40) follows from the equalities (2.19), (3.13′) and (3.14), respectively from the equalities (2.9) și (2.10) in (Vălcan, 1).

42) Equality (3.41) follows from the equalities (3.39) or (3.40).

43) Equality (3.42) follows from the equalities (2.19), (2.22), (3.36) and (3.39).

44) Equality (3.43) follows from the equalities (2.19), (2,22), (3.37) and (3.40), respectively from the equality (2.11) in (Vălcan, 2016).

45) Equality (3.44) follows from the equalities (3.42) or (3.43).

46) Equality (3.45) follows from the equalities (2.23), (2.19), (3.39) and (3.36).

47) Equality (3.46) follows from the equalities (2.23), (2,19), (3.40) and (3.37), respectively from the equality (2.12) in (Vălcan, 2016).

48) Equality (3.47) follows from the equalities (3.46) or (3.47).

49) Equality (3.48) follows from the equalities (2.23) and (3.18).

50) Equality (3.49) follows from the equalities (2.23) and (3.19).

51) Equality (3.50) follows from the equalities (3.48) or (3.49).

52) Equality (3.51) follows from the equalities (2.22) and (3.21).

53) Equality (3.52) follows from the equalities (2.22) and (3.22).

54) Equality (3.53) follows from the equalities (3.50) or (3.51).

Of course, and here, as in the previous work (Vălcan, 2022) we have presented only the ideas of proof, their verifications being immediate, we leave them to the reader.

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